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Electromagnetic corrections in hadronic processes

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Abstract. In many applications of chiral perturbation theory, one has to purify physical matrix elements from electromagnetic effects. On the other hand, the splitting of the Hamiltonian into a strong and an electromagnetic part cannot be performed in a unique manner, because photon loops generate ultraviolet divergences. In the present article, we propose a convention for disentangling the two effects: one matches the parameters of two theories – with and without electromagnetic interactions – at a given scale μ_1, referred to as the matching scale. This method enables one to analyse the separation of strong and electromagnetic contributions in a transparent manner. We first study in a Yukawa-type model the dependence of strong and electromagnetic contributions on the matching scale. In a second step, we investigate this splitting in the linear sigma model at one-loop order, and consider in some detail the construction of the corresponding low-energy effective Lagrangian, which exactly implements the splitting of electromagnetic and strong interactions carried out in the underlying theory. We expect these model studies to be useful in the interpretation of the standard low-energy effective theory of hadrons, leptons and photons.

1 Introduction

A systematic approach to take into account electromagnetic corrections in low-energy processes is based on chiral perturbation theory (ChPT), the low-energy effective theory of the standard model in the hadron sector. A general procedure for constructing this effective theory in the meson sector has been proposed by Urech [1]; see also [2]. A substantial number of papers has dealt with extensions of the method and with applications. In particular, Urech's approach has been generalized to include baryons [3] and leptons [4–6]. Numerical estimates of the electromagnetic low-energy constants (LECs) have been provided as well, based on different techniques (specific models, resonance saturation, sum rules) [7–11]. The effective Lagrangian with virtual photons has been used to study isospin-breaking corrections in the meson and baryon sectors (see, e.g., [3,12,13]), including hadronic atoms [14]. As the latest interesting developments, we mention the evaluation of isospin-breaking corrections in radiative τ decays, which is relevant for the analysis of the anomalous magnetic moment of the muon [15], and the construction of the chiral Lagrangian in the intrinsic parity odd sector at $O(e^2p^4)$; see [16]. In this last reference, electromagnetic corrections to $\pi^0 \to \gamma\gamma$ were evaluated as well.

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Despite these works, we believe that there is still room for improvement in the understanding of the presently used method to calculate electromagnetic corrections at low energies. To illustrate what we have in mind, consider the decay $\eta \to 3\pi$ in the framework of QCD [17]. The amplitude for this decay is proportional to $1/Q^2$, where

$$Q^2 = \frac{m_s^2 - \hat{m}^2}{m_d^2 - m_u^2}$$

denotes a ratio of quark masses in pure QCD. One approach is to use the measured decay width $\Gamma_{\eta\to 3\pi}$ for a determination of the quantity Q^2. On the other hand, one may as well evaluate Q^2 from the meson mass ratio

$$Q^2 = \frac{m_K^2}{m_\pi^2} \frac{m_K^2 - m_\pi^2}{\left(m_{K^0}^2 - m_{K^+}^2\right)_{\rm QCD}} \left(1 + O\left(m_{\rm quark}^2\right)\right),$$

and predict the width. In this manner, the mass difference of the kaons in pure QCD shows up. In order to determine this difference, one has to properly subtract the contributions from electromagnetic interactions to the kaon masses [18]. Here one encounters a problem: due to ultraviolet divergences generated by photon loops, the splitting of the Hamiltonian of QCD + γ into a strong and an electromagnetic piece is ambiguous. The calculation of $\left(M_{K^+}^2 - M_{K^0}^2\right)_{\rm QCD}$ in the effective theory must therefore reflect this ambiguity as well. An analogous problem occurs whenever one wants to extract hadronic quantities from matrix elements which are contaminated with electromagnetic contributions.

A problem of this type does not seem to appear in some of the calculations of radiative corrections in ChPT; see e.g. the calculation of pionic beta decay in [5]. One starts from an effective Lagrangian $\mathcal{L}_{\text{eff}}$ that contains strong and electromagnetic couplings, and evaluates physical processes in terms of these. The meson masses that occur in these calculations may be identified with the physical ones, and need not be split into a strong and an electromagnetic piece. However, at the end of the day, for a calculation of the matrix element, one needs a value for the remaining couplings involved. It is clear that, in principle, these can be determined from the underlying theory, if the effective theory is constructed properly. Since in that theory, there does exist an ambiguity as to what is an electromagnetic and what is a strong effect, the ambiguity must also reside in the couplings. Estimates of their sizes should therefore take into account this fact.

One is confronted with two separate issues here. The first one is a proper definition of strong and electromagnetic contributions in a given theory. The second, separate point concerns the construction of the corresponding effective low-energy Lagrangian. In an early stage these points were mentioned in [7]. In [8], Bijnens and Prades have evaluated several of the electromagnetic LECs by applying a combined approach, which uses the extended Nambu–Jona–Lasinio model (perturbative QCD and factorization) to evaluate long-distance (short-distance) contributions in the convolution integrals that determine these LECs. It is pointed out that some of these constants depend on the gauge and scale of the underlying theory. Explicit calculations are then carried out in the Feynman gauge. In [10], the dependence of the electromagnetic LECs on the QCD scale and on the gauge parameter is studied as well. A representation of the LECs in the form of a convolution of the pertinent QCD correlators with the photon propagator has been exploited for estimates of their size.

In our article, we take up these discussions. The final aim is
(i) to investigate the problem of electromagnetic corrections in QCD+γ, in the sense that the generating functional of Green functions of scalar, vector and axial vector currents is extended to include radiative corrections at order α, and
(ii) to construct the relevant effective theory at low energies, taking into account the ambiguities mentioned. It may be that the effective Lagrangian constructed some time ago by Urech [1] stays put. However, the LECs occurring in there certainly need a refined interpretation. Due to the complexity of the problem, we found it useful to investigate the issue – as a first step – in the framework of field-theoretical models which allow a perturbative analysis to be made. For this reason, we concentrate in the following on two models: first, on a theory with Yukawa interactions between fermions and scalar particles. This simple theory allows one to illustrate the separation of electromagnetic effects in a clear manner. In order to also investigate the transition to the relevant effective low-energy theory, we consider the linear σ-model (LσM) in its broken phase, with electromagnetic interactions added. Three different scales occur in these investigations: μ, the renormalization scale in the underlying theory; μ_{eff}, the one in the effective theory, and μ_1, the matching scale.

The scales μ and μ_{eff} have the standard interpretation. At the matching scale μ_1, the parameters in the full theory agree with those in the theory where the electromagnetic interactions are switched off, in a manner to be specified later in this article. The calculations, which we explicitly carry out in the framework of the loop expansion, allow us to illustrate the salient features of the electromagnetic corrections to processes that occur through the interactions of non-electromagnetic origin (called *strong interactions* for brevity in the following), and to illuminate the role of the three scales just mentioned [19].

The plan of this paper is as follows. In Sect. 2, we discuss our prescription for the splitting of strong and electromagnetic interactions in the Yukawa model, and analyse the ambiguity of such a splitting. The same questions are dealt with in Sects. 3, 4 and 5 within the linear sigma model in the spontaneously broken phase. In Sect. 6, we study the splitting in the corresponding low-energy effective theory. Comparing the quantities calculated in LσM and in the effective theory, we provide explicit expressions for some of the low-energy constants. Using these expressions, we discuss the dependence of the parameters of the effective theory on the matching scale, as well as on the running scale and on the gauge parameter of the underlying theory. In Sect. 7, we compare our results with the work of Moussallam [10]. Section 8 contains a summary and concluding remarks. The appendices collect some notation and useful formulae.

2 Separating strong and electromagnetic effects

2.1 Notation

We first illustrate in a Yukawa-type model the splitting of strong and electromagnetic interactions. While this theory does not describe the real world, the characteristic features of having several couplings in the theory are illustrative. The Lagrangian describes interactions between fermions, a scalar field and photons. The scalar field generates what we call here strong interactions. For simplicity, we consider the case of two couplings, g and e. The first one describes the interaction of the scalar field with the fermions, and e denotes the electric charge. Other couplings, e.g. the quartic self-interaction of the scalar field, will then arise through quantum fluctuations. In order to avoid vacuum diagrams (where the scalar field disappears in the vacuum), which render the renormalization more complicated, we equip the fermions and the scalars with an internal degree of freedom that we call *colour* for simplicity. The Lagrangian is

$$\mathcal{L}_Y = \bar{\Psi}\left([\mathrm{i}\not{D} - \mathcal{M}]\cdot 1_c + g\boldsymbol{\phi}\cdot 1_f\right)\Psi + \frac{1}{4}\left\langle \partial_\mu\boldsymbol{\phi}\,\partial^\mu\boldsymbol{\phi}\right\rangle_c - \frac{M^2}{4}\left\langle\boldsymbol{\phi}^2\right\rangle_c - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2 + \mathcal{L}_{\rm ct}. \quad (2.1)$$

Here $\mathcal{L}_{\rm ct}$ stands for the counterterms that render the generating functional finite at one-loop order. We use the following notation for the fermion and scalar fields:

$$\Psi \doteq \Psi_q^n\,; \quad q,n = 1,2\,; \quad \boldsymbol{\phi} \doteq \tau^a\phi^a\,, \quad (2.2)$$

where τ^a denote the Pauli matrices. We refer to $q\,(n)$ as flavour (colour) indices, respectively, and $\langle A\rangle_c$ denotes the colour trace of A. The unit matrices in the flavour (colour) space are denoted by $1_f \doteq \delta_{sq}$ ($1_c \doteq \delta^{nm}$), and e.g. $\bar{\Psi}\boldsymbol{\phi}\cdot 1_f\Psi$ stands for $\bar{\Psi}_q^n\tau^a_{nm}\phi^a\Psi_q^m$, etc. Further, A_μ denotes the photon field, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The quantity ξ stands for the gauge parameter ($\xi = 1$ corresponds to the Feynman gauge). The covariant derivative of the fermion field is defined as

$$D_\mu = \partial_\mu\cdot 1_f - \mathrm{i}eQA_\mu\,, \quad (2.3)$$

and $\mathcal{M}$ (M) stands for the fermion mass matrix (mass of the scalar field). The quantities Q and $\mathcal{M}$ are 2×2 matrices in flavour space,

$$Q = \frac{1}{3}\begin{pmatrix}2 & 0\\ 0 & -1\end{pmatrix} \doteq \begin{pmatrix}Q_1 & 0\\ 0 & Q_2\end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}. \quad (2.4)$$

Finally, eQ_q denotes the charge of the fermion q.

2.2 Renormalization

We consider the generating functional

$$\mathrm{e}^{\mathrm{i}Z_Y} = N\int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\mathcal{D}\boldsymbol{\phi}\mathcal{D}A_\mu \times \exp\left\{\mathrm{i}\int \mathrm{d}x\left[\mathcal{L}_Y + \bar{\eta}\Psi + \bar{\Psi}\eta + f^a\phi^a\right]\right\}. \quad (2.5)$$

Here, η and f^a are external sources for the fermion and for the scalar fields, and N is a normalization factor, chosen such that Z_Y vanishes in the absence of external fields. For the renormalization, we choose the modified minimal subtraction ($\overline{\rm MS}$) scheme. The generating functional Z_Y at one loop can be made finite by the following choice of ultraviolet divergent counterterms:[1]

$$\mathcal{L}_{\rm ct} = \lambda(\mu)\sum_{i=1}^{12}\sigma_i O_i\,, \quad (2.6)$$

where

$$\lambda(\mu) = \frac{\mu^{d-4}}{16\pi^2}\left(\frac{1}{d-4} - \frac{1}{2}\left(\Gamma'(1) + \ln 4\pi\right)\right). \quad (2.7)$$

Table 1. Counterterm Lagrangian: operator basis and the β-functions

i	O_i	σ_i
1	$\bar{\Psi}\left[\mathrm{i}\not{D} - \mathcal{M}\right]\cdot 1_c\Psi$	$3g^2$
2	$\bar{\Psi}Q\left[\mathrm{i}\not{D} - \mathcal{M}\right]Q\cdot 1_c\Psi$	$2\xi e^2$
3	$\bar{\Psi}\mathcal{M}\cdot 1_c\Psi$	$9g^2$
4	$\bar{\Psi}Q\mathcal{M}Q\cdot 1_c\Psi$	$-6e^2$
5	$\bar{\Psi}\boldsymbol{\phi}\cdot 1_f\Psi$	$2g^3$
6	$\bar{\Psi}\boldsymbol{\phi}\cdot Q^2\Psi$	$(6+2\xi)e^2g$
7	$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$	$\frac{80}{27}e^2$
8	$\frac{1}{2}\left\langle\boldsymbol{\phi}^2\right\rangle_c\left\langle\mathcal{M}^2\right\rangle_f$	$-24g^2$
9	$\frac{1}{2}\left\langle\partial_\mu\boldsymbol{\phi}\partial^\mu\boldsymbol{\phi}\right\rangle_c$	$8g^2$
10	$\frac{1}{4}\left\langle\boldsymbol{\phi}^2\right\rangle_c^2$	$-8g^4$

The operator basis O_i and the β-functions σ_i in (2.6) are displayed in Table 1.

In the language used here, the couplings g, e and the masses m_q are the running ones – we do not, however, indicate this fact with an index attached to these (or other running) parameters, in order to avoid flooding of the text with unnecessary symbols.

2.3 The physical mass

As a first application, we evaluate the physical mass of the fermion fields, given by the position of the pole in the propagator. Denoting these masses by M_q, we find

$$M_q = m_q\left[1 + \frac{3}{16\pi^2}\left(3g^2 - 2e^2Q_q^2\right)\ln\frac{m_q}{\mu} + A_1g^2 + A_2Q_q^2e^2\right] + O\left(g^4, e^2g^2, e^4\right)\,, \quad (2.8)$$

where

$$A_1 = \frac{3}{16\pi^2}\int_0^1 \mathrm{d}x(2-x)\ln\left(x^2 + (1-x)M^2/m_q^2\right)\,,$$
$$A_2 = \frac{1}{4\pi^2}\,. \quad (2.9)$$

The physical masses become scale independent, provided that the masses m_q run properly with the scale,

$$\mu\frac{\mathrm{d}m_q}{\mathrm{d}\mu} = \frac{3}{16\pi^2}\left(3g^2 - 2e^2Q_q^2\right)m_q + O\left(g^4, e^2g^2, e^4\right). \quad (2.10)$$

The scale dependence of g, e and of M^2 is a one-loop effect and therefore does not matter in the present context. Once

[1] We tame ultraviolet as well as infrared divergences with dimensional regularization. As usual, d denotes the dimension of space-time, and μ is the renormalization scale

the running mass m_q is known at some scale, the physical mass M_q is fixed in terms of the coupling constants g, e and of m_q, M at this order in the perturbative expansion; see below.

We now discuss the splitting of the physical masses into a strong and an electromagnetic part. A first choice might be to identify those parts of (2.8) which are proportional to g^2 (e^2) as the strong (electromagnetic) contributions to the mass. However, this identification has the disadvantage that the so defined strong piece runs with e^2 as well; see (2.10). For this reason, we define the splitting procedure as follows. We divide the mass into a piece that one would calculate in a theory with no electromagnetic interactions, and a part proportional to e^2,

$$M_q = \bar{M}_q + e^2 M_q^1 + O\left(e^4\right) . \tag{2.11}$$

Here and below, barred quantities refer to the theory at $e = 0$. The first term on the right-hand side is

$$\bar{M}_q = \bar{m}_q \left[1 + \frac{9\bar{g}^2}{16\pi^2} \ln \frac{\bar{m}_q}{\mu} + A_1 \bar{g}^2\right] + O\left(\bar{g}^4\right) . \tag{2.12}$$

This part is scale independent by itself, provided that the mass $\bar{m}_q$ runs according to

$$\mu \frac{\mathrm{d}\bar{m}_q}{\mathrm{d}\mu} = \frac{9}{16\pi^2} \bar{g}^2 \bar{m}_q + O\left(\bar{g}^4\right) . \tag{2.13}$$

The scale dependence of $\bar{g}$ does not matter at this order. The relation (2.13) shows that one has to fix a boundary condition in order to determine $\bar{M}_q$. As a natural condition, we choose the running mass $\bar{m}_q$ to coincide with the running mass m_q in the full theory at the scale $\mu = \mu_1$,

$$\bar{m}_q(\mu; \mu_1) = m_q(\mu_1) \left[1 + \frac{9\bar{g}^2}{16\pi^2} \ln \frac{\mu}{\mu_1}\right] + O\left(\bar{g}^4\right). \tag{2.14}$$

The electromagnetic part $e^2 M_q^1$ is obtained by evaluating the difference $M_q - \bar{M}_q$. Identifying g with $\bar{g}$ at this order, we finally have

$$\begin{aligned} \bar{M}_q &= \bar{m}_q(\mu; \mu_1) \left[1 + \frac{9\bar{g}^2}{16\pi^2} \ln \frac{\bar{m}_q}{\mu} + A_1 \bar{g}^2\right] + O\left(\bar{g}^4\right) , \\ M_q^1 &= -\bar{m}_q(\mu; \mu_1) \left[\frac{6}{16\pi^2} \ln \frac{\bar{m}_q}{\mu_1} - A_2\right] Q_q^2 + O\left(\bar{g}^2\right) . \end{aligned} \tag{2.15}$$

This splitting has the desired properties: the strong and the electromagnetic part are scale independent. On the other hand, as is explicitly seen in the contribution proportional to e^2, the splitting does depend on the *matching scale* μ_1. Indeed, one has at this order

$$\mu_1 \frac{\mathrm{d}\bar{M}_q}{\mathrm{d}\mu_1} = -\mu_1 \frac{\mathrm{d}\left[e^2 M_q^1\right]}{\mathrm{d}\mu_1} = -\frac{6e^2 Q_q^2}{16\pi^2} \bar{M}_q . \tag{2.16}$$

In other words, both terms in the splitting depend on the scale μ_1. This scale dependence is of order e^2 in the approximation considered. The sum M_q is of course independent of the matching scale.

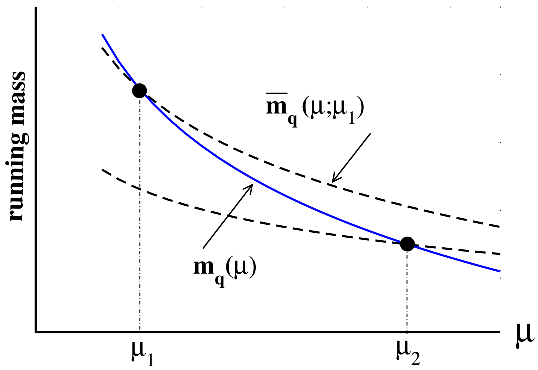

Fig. 1. The matching condition (2.14). The solid line represents the running of the mass m_q in the full theory according to (2.10), whereas the dashed lines display the running of $\bar{m}_q$ according to (2.13). The scales $\mu_{1,2}$ refer to matching scales, where $\bar{m}_q$ is made to agree with m_q

The dependence of the splitting on the scale μ_1 originates in the different running of the masses in the full theory and in the approximation when $e = 0$. In Fig. 1, we illustrate the matching condition (2.14). The solid line refers to the running of the mass m_q in the full theory, whereas the dashed lines represent the running of $\bar{m}_q$. Because, for a fixed value of the scale μ, the running mass $\bar{m}_q$ depends on the matching scale chosen, the mass $\bar{M}_q$ does so as well.

The splitting of the pole mass into a piece at $e = 0$ and a part proportional to e^2 can in general be performed from knowledge of the relevant β-functions of the masses and of the coupling constants to any order in the perturbative expansion; see below.

2.4 Splitting of the running masses

We have discussed the splitting of the physical masses into a strong and an electromagnetic piece. A similar splitting may be considered for the running masses themselves. Indeed, consider the matching condition (2.14). Expressing $m_q(\mu_1)$ through the running mass at scale μ gives at one-loop order

$$m_q(\mu) = \bar{m}_q(\mu; \mu_1) \left[1 - \frac{6e^2 Q_q^2}{16\pi^2} \ln \frac{\mu}{\mu_1}\right] . \tag{2.17}$$

This result is the analogue of the relation (2.11) for the physical masses. It shows that the splitting of the running masses into a part that runs with the strong interaction alone, and a piece proportional to e^2, depends on the matching scale; see Fig. 1.

This ambiguity in the splitting also occurs in QCD for the quark masses. At lowest order in the strong coupling g_s, the ambiguity in the mass of the q-quark is

$$\Delta \bar{m}_q = -\frac{3\alpha Q_q^2}{2\pi} \bar{m}_q \ln \frac{\mu_2}{\mu_1} . \tag{2.18}$$

In the case of the up quark (down quark), a change in scale by a factor two changes the value of m_u (m_d) by $1^0/_{00}$

($0.25^0/_{00}$). How does this affect e.g. the proton–neutron mass difference? We consider the first two terms in the quark mass expansion in pure QCD,

$$M_p = M_0 + B^u m_u + B^d m_d + \dots\,,$$
$$M_n = M_0 + B^d m_u + B^u m_d + \dots \quad (2.19)$$

Here, M_0 denotes the nucleon mass in the chiral limit, and $B^{u,d}$ stand for nucleon matrix elements of quark bilinears. The ellipses denote higher order terms in the quark mass expansion. For the proton–neutron mass difference, we obtain

$$M_p - M_n = \left(B^d - B^u\right)(m_d - m_u) + \dots\,, \quad (2.20)$$

where the ellipsis denotes higher order terms in the quark mass expansion. The ambiguity in this splitting is

$$\Delta(M_p - M_n) \simeq \left(B^d - B^u\right)(\Delta m_d - \Delta m_u) + \left(\Delta B^d - \Delta B^u\right)(m_d - m_u)\,. \quad (2.21)$$

The second term on the right-hand side is induced by the analogous ambiguity in the splitting of the strong coupling constant g_s. It is an effect of second order in isospin violation, and we drop it here. As a result, we have

$$\Delta(M_p - M_n) \simeq -\frac{3\alpha}{18\pi}\frac{m_d/m_u - 4}{m_d/m_u - 1}(M_p - M_n)\ln\frac{\mu_2}{\mu_1} \simeq 10^{-3}\cdot(M_p - M_n)\ln\frac{\mu_2}{\mu_1} \quad (2.22)$$

for $m_d/m_u \simeq 1.75$. A change in the scale μ_1 by a factor 2 therefore changes the mass difference by less than $1^0/_{00}$, a negligible effect. (See [21] for the analogous discussion concerning the mass M_0.)

2.5 Renormalization group and the splitting procedure

The above examples illustrate the salient features of purifying mass parameters from electromagnetic effects. One may wonder whether there is a way to split e.g. the pole mass in a unique manner. The reason why this is not the case is the following. In the Yukawa model considered here, the pole mass is proportional to m_q, which itself depends on the scale μ. In order to compare this mass with the corresponding quantity at $e = 0$, one has to compare two quantities that run differently, $\bar{m}_q$ and m_q. This running is itself a one-loop effect. Beyond the tree-level approximation, the inherent ambiguity therefore will show up unavoidably.

The proper tools to perform the splitting in general are the β-functions of the masses and of the coupling constants. For illustration, let us consider a theory which has only the following parameters: strong and electromagnetic couplings g, e and a mass m. We do not specify the physical content of this theory, since it does not play any role here, and assume that the renormalization group equations (RGE) read

$$\mu\frac{\mathrm{d}g}{\mathrm{d}\mu} = \beta_g(g,e) = \beta_g^{(0)}(g) + e^2\beta_g^{(1)}(g) + O\left(e^4\right)\,,$$
$$\mu\frac{\mathrm{d}e}{\mathrm{d}\mu} = \beta_e(g,e) = e^3\beta_e^{(0)}(g) + e^5\beta_e^{(1)}(g) + O\left(e^7\right)\,, \quad (2.23)$$
$$\mu\frac{\mathrm{d}m}{\mathrm{d}\mu} = \gamma(g,e)\,m = \left[\gamma^{(0)}(g) + e^2\gamma^{(1)}(g) + O\left(e^4\right)\right]m\,.$$

The RGE in the theory with no virtual photons are obtained from (2.23) by retaining for g and m only the first term in the expansion in e^2,

$$\mu\frac{\mathrm{d}\bar{g}}{\mathrm{d}\mu} = \beta_g^{(0)}(\bar{g})\,, \qquad \mu\frac{\mathrm{d}\bar{m}}{\mathrm{d}\mu} = \gamma^{(0)}(\bar{g})\,\bar{m}\,, \quad (2.24)$$

where bars indicate quantities defined in the theory with no virtual photons. The matching condition sets the parameters $(g,\bar{g})$ and $(m,\bar{m})$ equal at the matching scale $\mu = \mu_1$. With this condition, the couplings and the masses can unambiguously be related to each other,

$$g(\mu) = \bar{g}(\mu;\mu_1)\left[1 + e^2(\mu)X_g(\bar{g},\mu,\mu_1) + O\left(e^4\right)\right]\,,$$
$$m(\mu) = \bar{m}(\mu;\mu_1)\left[1 + e^2(\mu)X_m(\bar{g},\mu,\mu_1) + O\left(e^4\right)\right]\,, \quad (2.25)$$

where the explicit expressions for $X_g,\ X_m$ can be obtained order by order in perturbation theory. The splitting of other quantities proceeds in an analogous manner.

3 Linear sigma model

In the remaining part of this paper, we consider the Higgs model in its spontaneously broken phase. The model also goes under the name *linear sigma model* (LσM); we stick to it in the following. In the absence of electromagnetic interactions, it exhibits an $O(4)$ symmetry, spontaneously broken to $O(3)$. The corresponding effective theory at low energies may be analysed with the Lagrangian used in ChPT, with low-energy constants that are fixed in terms of the couplings of the LσM [22,23]. Here, we extend these investigations to incorporate also electromagnetic interactions. In particular, in this and in the following two sections, we evaluate several quantities (pole masses, coupling constants and vector current matrix elements) at one loop, and discuss the disentangling of electromagnetic effects. In Sect. 6, we then consider the corresponding low-energy effective theory and work out the low-energy expansion of the results obtained in the LσM, which amounts to matching certain combinations of LECs in the effective theory. This will allow us to investigate the scale and gauge dependence of these LECs.

Although the linear sigma model does not qualify as a candidate for the strong interactions [22], we expect that many features of its effective low-energy theory are very similar to the one of QCD + γ. For simplicity, we refer in the following to the linear sigma model at $e = 0$ as the *strong (underlying) theory.*

We start the discussion with the construction of the Lagrangian at one-loop order.

3.1 The Lagrangian

We couple the four real scalar fields ϕ^A in the linear sigma model to external vector and axial vector fields and incorporate electromagnetic interactions,

$$\mathcal{L}_\sigma = \mathcal{L}_0 + \mathcal{L}_{\rm ct}\,,$$

$$\mathcal{L}_0 = \frac{1}{2}\left(d_\mu\phi\right)^{\rm T} d^\mu\phi + \frac{m^2}{2}\,\phi^{\rm T}\phi - \frac{g}{4}\left(\phi^{\rm T}\phi\right)^2 + c\phi^0 + \frac{\delta m^2}{2}\,(Q\cdot\phi)^{\rm T}(Q\cdot\phi) - \frac{\delta g}{2}\,\left(\phi^{\rm T}\phi\right)(Q\cdot\phi)^{\rm T}(Q\cdot\phi) - \frac{1}{4}\,F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\,(\partial_\mu A^\mu)^2\,, \quad (3.1)$$

where

$$d^\mu\phi = \partial^\mu\phi + \Gamma^\mu\cdot\phi\,, \quad (\Gamma^\mu\cdot\phi)_A = \Gamma^\mu_{AB}\,\phi^B\,, \quad (3.2)$$

$$\Gamma^\mu = F^\mu + eA^\mu Q\,, \quad \Gamma^{\mu\nu} = \partial^\mu\Gamma^\nu - \partial^\nu\Gamma^\mu + [\Gamma^\mu,\Gamma^\nu]\,,$$

and where the external vector and axial vector fields are collected in the antisymmetric matrix F_μ,

$$F^{0i}_\mu = a^i_\mu\,, \qquad F^{ij}_\mu = -\epsilon^{ijk}v^k_\mu\,. \quad (3.3)$$

The notation for $A_\mu, F_{\mu\nu}$ and for ξ is the same as in the previous section. The only non-zero entries of the 4×4 charge matrix Q_{AB} are[2] $Q_{12} = -Q_{21} = -1$. In our metric, the spontaneously broken phase occurs at $m^2 > 0$. Since the electromagnetic interactions break isospin symmetry, we have explicitly introduced the isospin-breaking terms $\sim \delta m^2, \delta g$ from the very beginning. The counterterms are collected in $\mathcal{L}_{\rm ct}$; see below. The symmetry breaking parameter c is considered to be of non-electromagnetic origin – it provides the Goldstone bosons with a mass also at $e = 0$.

The evaluation of the masses and of the current matrix elements will be performed in the loop expansion. In order to be consistent, on the one hand, with our assumption that isospin breaking has a purely electromagnetic origin and, on the other hand, with ChPT counting rules, we will furthermore rely on the following counting for the symmetry breaking parameters,

$$\delta m^2 \simeq O\left(e^2\right)\,, \quad \delta g \simeq O\left(e^2\right)\,; \quad e^2, c \simeq O\left(p^2\right)\,. \quad (3.4)$$

3.2 Renormalization

One-loop divergences are removed by the following counterterms,

$$\mathcal{L}_{\rm ct} = \lambda(\mu)\sum_{i=1}^{8}\beta_i P_i + O\left(e^4\right)\,, \quad (3.5)$$

where the divergent quantity $\lambda(\mu)$ is displayed in (2.7). The operator basis and the β-functions are collected in Table 2. We use the $\overline{\rm MS}$ scheme to eliminate the divergences in the Green functions. The parameters of the theory obey the following renormalization group equations,

$$\mu\frac{\rm d}{{\rm d}\mu}\begin{pmatrix} m^2 \\ c \\ \delta m^2 \end{pmatrix} = \hat\gamma \begin{pmatrix} m^2 \\ c \\ \delta m^2 \end{pmatrix}\,, \qquad \mu\frac{\rm d}{{\rm d}\mu}\begin{pmatrix} g \\ \delta g \\ e \end{pmatrix} = \hat\beta\,, \quad (3.6)$$

[2] Our notation for the charge matrices is summarized in Appendix A

Table 2. Counterterm Lagrangian in the LσM with virtual photons and external fields, in any gauge: operator basis and the β-functions

i	P_i	β_i
1	$\frac{1}{2}\,\phi^{\rm T}\phi$	$-12gm^2 - 4m^2\delta g - 4g\delta m^2$
2	$-\frac{1}{4}\,\left(\phi^{\rm T}\phi\right)^2$	$-24g^2 - 8g\delta g$
3	$\frac{1}{2}\,(Q\cdot\phi)^{\rm T}(Q\cdot\phi)$	$2m^2e^2\xi - 16m^2\delta g - 4g\delta m^2$
4	$-\frac{1}{2}\,\left(\phi^{\rm T}\phi\right)(Q\cdot\phi)^{\rm T}(Q\cdot\phi)$	$2ge^2\xi - 40g\delta g$
5	$\frac{1}{2}\,(Q\cdot d_\mu\phi)^{\rm T}\,(Q\cdot d^\mu\phi)$	$-6e^2 + 2e^2\xi$
6	$\frac{1}{8}\,{\rm tr}\,\Gamma_{\mu\nu}\Gamma^{\mu\nu}$	$\frac{2}{3}$
7	$([F_\mu,Q]\cdot\phi)^{\rm T}\,(Q\cdot d^\mu\phi)$	$-3e^2 + 2e^2\xi$
8	$([F_\mu,Q]\cdot\phi)^{\rm T}([F^\mu,Q]\cdot\phi)$	$\frac{3e^2}{4}\,(\xi - 1)$

where

$$\hat\gamma = \frac{1}{16\pi^2}\begin{pmatrix} 12g + 4\delta g & 0 & 4g \\ 0 & 0 & 0 \\ -6e^2 + 16\delta g & 0 & 4g \end{pmatrix}\,,$$

$$\hat\beta = \frac{1}{16\pi^2}\begin{pmatrix} 24g^2 + 8g\delta g \\ -6ge^2 + 40g\delta g \\ \frac{1}{3}\,e^3 \end{pmatrix}\,. \quad (3.7)$$

The RGE in the isospin symmetric case can be obtained by setting $e = \delta g = \delta m^2 = 0$.

The following remarks are in order.

(1) The operator P_7, which contributes to the renormalization of the matrix element of the charged vector current, has no counterpart in the tree Lagrangian. This implies that, in the $\overline{\rm MS}$ scheme, the charged components of the vector form factor become scale (and gauge) dependent in the presence of electromagnetic interactions. This merely reflects the fact that the charged current is not an observable quantity for $e \neq 0$. An analogous situation occurs in QCD + γ.

(2) On the other hand, if one calculates the matrix element of the charged current in the effective theory, it is apparently scale independent, and in general exhibits a different gauge dependence. In order to reconcile these two ways of calculation, the electromagnetic LECs in the effective theory must depend on the running scale of the underlying theory and on the gauge parameter [7,8,10].

(3) There is an essential difference between the contact term P_6 that arises in the renormalization of the theory at $e = 0$, and the operators P_7, P_8. None of them have counterparts at tree level. However, whereas P_6 at $e = 0$ contains only external sources and does not contribute to S-matrix elements, the operators $P_{7,8}$ carry dynamical fields along with the external sources, and therefore do show up in physical matrix elements.

4 Masses and couplings in the LσM

In this section, we evaluate the charged and neutral pion masses in the spontaneously broken phase of the linear sigma model to one loop. Further, we discuss their splitting into a strong and an electromagnetic part. In order to keep track of the notation, we found it useful to provide in Appendix F a separate glossary for the various mass parameters used.

4.1 Tree level

For $m^2 > 0$, the potential has a minimum at $\phi^{\mathrm{T}} = (v_0, \mathbf{0}) \neq 0$. After the shift $\phi^0 = \sigma + v_0$, one may directly read off the expression for the vacuum expectation value v_0 and for the masses at tree level. In particular, v_0 satisfies the equation

$$g v_0^2 - m^2 - \frac{c}{v_0} = 0\,, \tag{4.1}$$

from which one has

$$v_0 = \frac{m}{\sqrt{g}} + \frac{c}{2m^2} + O\left(p^4\right). \tag{4.2}$$

The pion and the sigma masses at tree level are

$$\begin{aligned} m_{\pi^0}^2 &= \frac{c}{v_0}\,, \\ m_{\pi^+}^2 &= m_{\pi^0}^2 - \delta m^2 + \delta g v_0^2\,, \\ m_\sigma^2 &= 2m^2 + 3m_{\pi^0}^2\,. \end{aligned} \tag{4.3}$$

4.2 One loop

The vacuum expectation value of the field ϕ^0 is evaluated in the standard manner: one performs the shift $\phi^0 = \sigma + v$, and calculates the one-point function of the σ field to one loop. The corresponding diagrams are depicted in Fig. 2. The requirement $\langle 0|\sigma|0\rangle = 0$ then determines v. We find

$$v = v_0 \left\{1 - \frac{g}{m_\sigma^2}\left(3L_\sigma + L_{\pi^0} + 2L_{\pi^+}\right)\right\} + O\left(p^4, \hbar^2\right), \tag{4.4}$$

where

$$L_X = \frac{m_X^2}{16\pi^2}\left\{\ln\frac{m_X^2}{\mu^2} - 1\right\}. \tag{4.5}$$

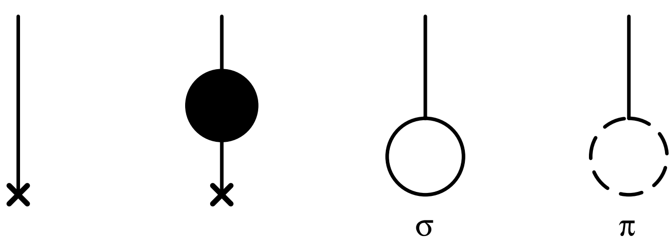

Fig. 2. The vacuum expectation value of the scalar field ϕ^0. Displayed are diagrams that occur at tree level and at one-loop order. The shaded blob denotes self-energy insertions. Counterterm contributions are not shown

Fig. 3. Self-energy of the pions at one loop. Counterterm contributions are not shown. Dashed lines correspond to $\pi^\pm$, π^0, solid lines to σ, and the wiggle line to the photon. The last diagram is absent for the neutral pion

The quantities m_X denote the tree-level masses in (4.3), and v_0 is the solution to (4.1). Let us make the following remarks. The expansions in the framework of the linear sigma model are twofold: expansions in $\hbar$ and in powers of the momenta, according to (3.4). As an example, the ratio v/v_0 takes the form $v/v_0 = a_0 + a_1 p^2 + a_2 p^4 + \ldots$, where the coefficients a_i are represented by a series expansion in $\hbar$. We have indicated this fact in (4.4) with the Landau symbol $O(p^4, \hbar^2)$. To make the notation easier, we often drop in the following the symbol $\hbar^2$ altogether. Furthermore, we will make use of the power counting convention (3.4), so that $O(e^2p^2, p^4)$ is written as $O(p^4)$. Finally, we drop terms of order e^4 in the calculations, and do not indicate this in the Landau symbols, except in the low-energy expansion of the pion masses.

To determine the pion masses, we evaluate the pole positions in the Fourier transform of the two-point functions $\left\langle 0|T\phi^i(x)\phi^i(0)|0\right\rangle$, $i = 1, 3$. The relevant diagrams are displayed in Fig. 3. We find

$$\begin{aligned} M_{\pi^0}^2 &= m_{\pi^0}^2\left\{1 + \frac{g}{m_\sigma^2}\left(V_0 + 2L_{\pi^+} - L_{\pi^0}\right)\right\} + O\left(e^4, p^6\right), \\ M_{\pi^+}^2 &= m_{\pi^+}^2\left\{1 + \frac{g}{m_\sigma^2}\left(V_0 + L_{\pi^0}\right)\right\} - e^2\left\{3L_{\pi^+} - \frac{m_{\pi^+}^2}{4\pi^2}\right\} \\ &\quad + \frac{g}{m_\sigma^2}\left(m_{\pi^0}^2 - m_{\pi^+}^2\right)V_1 + \delta g V_2 + O\left(e^4, p^6\right), \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} V_0 &= (3 + 2y)\,L_\sigma - \frac{m_\sigma^2}{48\pi^2}(3 + 7y)\,, \\ V_1 &= L_{\pi^0} + 4L_{\pi^+} + (1 - 4y)L_\sigma + \frac{5m_{\pi^0}^2}{24\pi^2}\,, \\ V_2 &= (2 + 3y)L_\sigma - \frac{m_{\pi^0}^2}{8\pi^2}\,; \quad y = \frac{m_{\pi^0}^2}{m_\sigma^2}\,. \end{aligned} \tag{4.7}$$

4.3 The matching scale μ_1

In order to split the physical parameters into a strong part and a piece proportional to e^2, it is most useful to consider such a splitting for the running parameters of the theory itself, as was discussed in Sect. 2. We denote the parameters in the original theory by $\bar{g}, \bar{m}$ and $\bar{c}$. First, we note that c is not running at this order, so the matching condition is simply $\bar{c} = c$. Concerning $\bar{m}, \bar{g}$, one has several choices for the matching, because, in the presence of electromagnetic interactions, there are additional parameters δm^2 and δg that enter the theory. We stick to the simplest possible choice, matching $\bar{g}$ and $\bar{m}^2$ to their counterparts in the full theory at a scale $\mu = \mu_1$. With this choice of the matching condition, and using the RGE (3.6), one may relate the masses and coupling constants in these two theories at any scale.

In order to render the matching formulae more compact, we introduce the following notation for the isospin-breaking couplings δm^2 and δg,

$$\delta g = e^2 g c_g \,, \quad \delta m^2 = e^2 m^2 c_m \,, \tag{4.8}$$

where the new couplings c_g and c_m are assumed to be independent of e at this order. This notation makes it evident that δm^2 and δg are considered to be proportional to e^2. Of course, the RGE (3.6) and (3.7) can be easily rewritten in terms of the parameters g, m^2, e, c_g and c_m. The analogues of the matching equations (2.25) are now

$$\begin{aligned} g(\mu) &= \bar{g}(\mu;\mu_1)\left\{1 + c_g \frac{e^2\bar{g}}{2\pi^2} \ln\frac{\mu}{\mu_1}\right\}, \\ m^2(\mu) &= \bar{m}^2(\mu;\mu_1)\left\{1 + (c_g + c_m)\frac{e^2\bar{g}}{4\pi^2}\ln\frac{\mu}{\mu_1}\right\}, \\ c &= \bar{c}\,. \end{aligned} \tag{4.9}$$

As a result, the parameters $\bar{g}$ and $\bar{m}$ of the purely strong theory depend on the choice of the matching scale μ_1 in the following manner:

$$\mu_1 \frac{\mathrm{d}\bar{g}}{\mathrm{d}\mu_1} = \frac{e^2 c_g}{2\pi^2}\,\bar{g}^2\,,\ \mu_1\frac{\mathrm{d}\bar{m}^2}{\mathrm{d}\mu_1} = \frac{e^2\bar{g}(c_g + c_m)}{4\pi^2}\bar{m}^2. \tag{4.10}$$

Due to gauge invariance, the running of e starts at e^3, and does not affect other parameters at $O(e^2)$ – for this reason, we neglect this running and consider in the following e^2 to be a fixed coupling constant.

4.4 Electromagnetic effects in the pion masses

The pion masses may now be split into an isospin symmetric part and electromagnetic contributions in the following manner. One starts from (4.6) and expresses the parameters g, m, c through the isospin symmetric couplings $\bar{g}, \bar{m}$ and $\bar{c}$ by use of (4.9). The isospin symmetric part of the masses is obtained by putting the electric charge to zero, and the part proportional to e^2 is given by the difference of the full and the isospin symmetric part. Next, we observe that the dependence on the electric charge in (4.9) is an effect of order $\hbar$. Therefore, to the accuracy considered here, the splitting (4.9) must be applied to the tree-level expressions only,

$$\begin{aligned} v_0 &= \bar{v}_0\left\{1 - C\ln\mu^2/\mu_1^2\right\} + O\left(p^4\right), \\ m_{\pi^0}^2 &= \bar{m}_\pi^2\left\{1 + C\ln\mu^2/\mu_1^2\right\} + O\left(p^6\right), \end{aligned} \tag{4.11}$$

where

$$C = (c_g - c_m)\frac{e^2\bar{g}}{16\pi^2}\,, \quad \bar{m}_\pi^2 = \frac{\bar{c}}{\bar{v}_0}\,. \tag{4.12}$$

Here, $\bar{v}_0$ satisfies (4.1) at $(g,m)\to(\bar{g},\bar{m})$. The μ_1-dependence is

$$\mu_1\frac{\mathrm{d}}{\mathrm{d}\mu_1}\left(\bar{m}_\pi^2, \bar{v}_0\right) = 2C\left(\bar{m}_\pi^2, -\bar{v}_0\right). \tag{4.13}$$

Finally, the splittings become

$$X = \bar{X} + e^2 X^1 + O\left(e^4\right); \quad X = M_{\pi^0}^2, M_{\pi^+}^2\,. \tag{4.14}$$

We will use this notation also below: with a barred quantity we denote an expression evaluated at $e = 0$, with $(g,m)\to(\bar{g},\bar{m})$.

We illustrate (4.14) for the pion mass. The barred quantity is the same for the neutral and for the charged pion mass,

$$\begin{aligned} \bar{M}_\pi^2 &\doteq \bar{M}_{\pi^0}^2 = \bar{M}_{\pi^+}^2 \\ &= \bar{m}_\pi^2\left\{1 + \frac{\bar{g}}{\bar{m}_\sigma^2}\left(\bar{V}_0 + \bar{L}_\pi\right)\right\} + O\left(p^6\right). \end{aligned} \tag{4.15}$$

The electromagnetic corrections are given by the difference $M_{\pi^{0,+}}^2 - \bar{M}_\pi^2$. For the neutral pion mass they are

$$\begin{aligned} e^2 M_{\pi^0}^{2,1} &= \frac{\bar{m}_\pi^2\bar{g}}{16\pi^2\bar{m}^2}\left(m_{\pi^+}^2\ln\frac{m_{\pi^+}^2}{\mu^2} - m_{\pi^0}^2\ln\frac{m_{\pi^0}^2}{\mu^2}\right) \\ &\quad + \bar{m}_\pi^2 C\left(\ln\frac{\mu^2}{\mu_1^2} - 1\right) + O\left(e^4, p^6\right). \end{aligned} \tag{4.16}$$

A similar expression holds for the charged pion mass.

The quantity $\bar{M}_\pi$ denotes the isospin symmetric part of the pion mass. It coincides neither with the neutral nor with the charged pion mass, and is independent of the running scale μ. It depends, however, on the scale μ_1 where the matching has been performed,

$$\mu_1\frac{\mathrm{d}}{\mathrm{d}\mu_1}\bar{M}_\pi^2 = 2C\bar{m}_\pi^2 + O\left(e^4, p^6\right). \tag{4.17}$$

As C is of order e^2, this scale dependence of the isospin symmetric part is of order p^4. The electromagnetic part $e^2 M_{\pi^0}^{2,1}$ has the same scale dependence, up to a sign, as a result of which the total mass is independent of μ_1.

5 Vector currents in the LσM

We now consider matrix elements of the charged and neutral vector currents in the framework of the linear sigma model. The result enables one to explicitly determine the dependence of some of the electromagnetic LECs on the scale of the underlying theory, and on the gauge parameter.

We set the external axial vector source to zero, $a_\mu^i(x) = 0$. The two-point function of the pion fields in the presence of the external vector source is given by

$$\begin{aligned} &\int \mathrm{d}x\mathrm{d}y\, \mathrm{e}^{\mathrm{i}p'x - \mathrm{i}py}\,\left\langle 0|T\phi^i(x)\phi^j(y)|0\right\rangle_c \\ &= \int \mathrm{d}x\mathrm{d}y\,\mathrm{e}^{\mathrm{i}p'x-\mathrm{i}py}\,\left\langle 0|T\phi^i(x)\phi^j(y)|0\right\rangle_c\Big|_{v^\lambda=0} \end{aligned} \tag{5.1}$$

$$- \int \mathrm{d}u \, \mathrm{e}^{\mathrm{i}(p'-p)u} v_\mu^k(u) \; g^{\mu\rho} \Gamma_\rho^{ijk}(p',p) + O\left(v^2\right) .$$

The residue of the vertex function Γ^μ_{ijk} contains the form factors,

$$\Gamma_\mu^{ijk}(p',p) = \frac{Z_i^{1/2}}{M_i^2 - p'^2} \, F_\mu^{ijk}(p',p) \, \frac{Z_j^{1/2}}{M_j^2 - p^2} . \quad (5.2)$$

On the mass-shell $p_i^2 = M_i^2$, we have

$$F_\mu^{ijk}(p',p) = (p'+p)_\mu F_+^{ijk}(t) + (p-p')_\mu F_-^{ijk}(t) ;$$
$$t = (p'-p)^2 . \quad (5.3)$$

In the above formulae, $M_1^2 = M_2^2$ and M_3^2 are the physical masses of charged and neutral pions, respectively. Further, for the wave function renormalization constants one has $Z_1 = Z_2 = Z_{\pi^+}$ and $Z_3 = Z_{\pi^0}$. At one loop, these quantities can be obtained by evaluating the diagrams in Fig. 3.

5.1 Charged current

The matrix element of the charged vector current is obtained from (5.3) at $i=2$, $j=3$, $k=1$. Here, we concentrate on the form factor $F_+(t) \doteq F_+^{231}(t)$. The relevant one-loop diagrams are displayed in Fig. 4. In the large m_σ limit, we obtain

$$F_+(t) = 1 - \frac{gt}{96\pi^2 m^2}\left(\ln\frac{m_{\pi^+}^2}{2m^2} + \frac{11}{6} - \frac{6}{t} J_c\left(t; m_{\pi^+}^2, m_{\pi^0}^2\right)\right)$$
$$- \frac{e^2}{64\pi^2}\left((3-2\xi)\ln\frac{m_{\pi^+}^2}{\mu^2} + 7\right)$$
$$+ e^2 \lambda_{\mathrm{IR}}\left(m_{\pi^+}\right)(3-\xi) + O\left(p^4\right) . \quad (5.4)$$

The loop function $J_c(t; m_1^2, m_2^2)$ and the infrared divergent part λ_{IR} are displayed in Appendix B. The form factor $F_+(t)$ is infrared divergent. This, however, does not pose an obstacle for carrying out the matching of the low-energy effective theory, since the infrared divergences in both theories have the same form. Note that $F_+(t)$ is scale and gauge dependent at $e \neq 0$,

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} F_+(t) = \frac{e^2}{32\pi^2}(3 - 2\xi) . \quad (5.5)$$

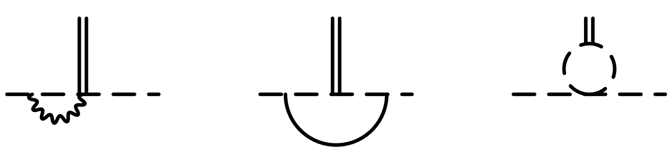

Fig. 4. One-loop contributions to the matrix element of the charged vector current in the linear sigma model. Counterterm contributions and external lines insertions are not shown. Double, solid, dashed and wiggle lines correspond to the external vector source, to the σ field, to (charged or neutral) pions and to photons, respectively

It is therefore not an observable quantity. The scale dependence is of purely ultraviolet origin, because loop diagrams are scale independent. Indeed, the scale dependence (5.5) is linked to entry 7 of Table 2 and could be obtained without doing an explicit evaluation of the loop diagrams.

5.2 Neutral current

In this subsection, we consider the matrix element of the neutral vector current, which is closely linked to the electromagnetic form factor of the pion. As we shall see, this quantity is scale dependent at $e \neq 0$, due to vacuum polarization effects.

The matrix element of the neutral current is defined as $F_0(t) \doteq F_+^{123}(t)$. Note that, due to the invariance of the theory under local $O(4)$ transformations (see the discussion in the following section), $F_-^{123}(t) = 0$. Introducing the function $\Phi(t) \doteq F_0(t)/F_0(0)$ which is normalized to unity at $t=0$, we obtain

$$\Phi(t) = 1 - \frac{gt}{96\pi^2 m^2}\left\{\ln\frac{m_{\pi^+}^2}{2m^2} + \frac{13}{6} + \sigma^2 K\left(t/m_{\pi^+}^2\right)\right\}$$
$$+ \frac{e^2}{16\pi^2}\left\{\left(\frac{t}{m_{\pi^+}^2} - 2\right) G\left(t/m_{\pi^+}^2\right) + \frac{\sigma^2}{3} K\left(t/m_{\pi^+}^2\right) - \frac{2}{9}\right.$$
$$\left. + 2\frac{\left(2m_{\pi^+}^2 - t\right) K\left(t/m_{\pi^+}^2\right) - t}{t - 4m_{\pi^+}^2}\left[32\pi^2\lambda_{\mathrm{IR}}(m_{\pi^+}) - 1\right]\right\}$$
$$+ O\left(p^4\right) ,$$
$$\sigma^2 = 1 - \frac{4m_{\pi^+}^2}{t} . \quad (5.6)$$

The loop functions G, K and the infrared divergent part λ_{IR} are displayed in Appendix B. (As a check on the calculation, we note that the form factors $F_+(t)$ and $F_0(t)$ coincide at $e=0$.) Note that $\Phi(t)$ is scale independent. On the other hand, the form factor at zero momentum transfer is

$$F_0(0) = 1 + \frac{e^2}{48\pi^2} \ln\frac{m_{\pi^+}^2}{\mu^2} . \quad (5.7)$$

It is straightforward to check that the correction term in (5.7) is due to vacuum polarization diagrams: as is well known, the contributions from the vertex correction and insertions in the external legs cancel at $t=0$. At one-loop order, the scale dependence of the form factor $F_0(t)$ is therefore

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} F_0(t) = -\frac{e^2}{24\pi^2} . \quad (5.8)$$

Also here, one can obtain this scale dependence without doing an explicit calculation. The only scale dependence that matters is that of the vacuum polarization operator at $t=0$, which is determined by β_6 from Table 2.

As a final remark, we mention that the splitting of strong and electromagnetic interactions at one loop can be unambiguously carried out in both the charged and the neutral current matrix elements. In other words, no μ_1-dependence arises in these quantities at this order. On the other hand, branch points are different in the full and in the strong part of the form factors. In particular, the loop function $K(t/\bar{m}_\pi^2)$ appearing in the strong part of the function $\Phi(t)$ defined by (5.6) has a branch point at $t = 4\bar{m}_\pi^2$, which does not coincide with the position of the physical branch point $t = 4m_{\pi^+}^2$. Let $\bar{m}_\pi^2 < m_{\pi^+}^2$. Then the strong form factor $\bar{\Phi}$ develops an imaginary part in the interval $4\bar{m}_\pi^2 \leq t \leq 4m_{\pi^+}^2$, where the full form factor is real. However, the imaginary part is of order e^2 in this interval, and the strong form factor differs from the full one only by terms of order e^2 also here.

6 Low-energy effective theory

6.1 Symmetries

As we mentioned in Sect. 3, the linear sigma model at $e = 0$ may be analysed at low energy with the effective Lagrangian of chiral $SU(2)_\mathrm{R} \times SU(2)_\mathrm{L}$ constructed a long time ago [22]. Here, we wish to extend the discussion to the case where the effects of virtual photons are included. These interactions break $O(4)$ symmetry. In order to apply the standard low-energy analysis which provides the structure of the effective theory, one enlarges the Lagrangian (3.1) in a manner proposed by Urech [1], such that the $O(4)$ symmetry is formally restored. The procedure goes as follows. First, the charge matrix Q in the Lagrangian (3.1) is promoted to space-time dependent spurion fields $Q_\mathrm{L}(x)$, $Q_\mathrm{R}(x)$. In matrix notation, to which we stick in this subsection, the expression for the covariant derivative becomes

$$d_\mu \Sigma = \partial_\mu \Sigma - \mathrm{i}(r_\mu + eQ_\mathrm{R}A_\mu)\Sigma + \mathrm{i}\Sigma(l_\mu + eQ_\mathrm{L}A_\mu), \quad (6.1)$$

where

$$\Sigma = \phi^0 + \mathrm{i}\tau^i\phi^i\,, \quad v_\mu = v_\mu^i\frac{\tau^i}{2}\,, \quad a_\mu = a_\mu^i\frac{\tau^i}{2}\,,$$
$$r_\mu = v_\mu + a_\mu\,, \quad l_\mu = v_\mu - a_\mu\,, \quad (6.2)$$

and where τ^i are the Pauli matrices. We consider charge spurions that are traceless, $\langle Q_\mathrm{R}\rangle = \langle Q_\mathrm{L}\rangle = 0$. The derivative part in the Lagrangian (3.1) is modified,

$$\frac{1}{2}(d_\mu\phi)^\mathrm{T} d^\mu\phi \to \frac{1}{4}\left\langle d_\mu\Sigma d^\mu\Sigma^\dagger\right\rangle\,,$$

where $\langle A\rangle$ denotes the trace of the matrix A. The symmetry breaking parts proportional to $(Q\cdot\phi)^\mathrm{T}Q\cdot\phi$ are replaced in an analogous manner,

$$(Q\cdot\phi)^\mathrm{T}Q\cdot\phi \to -\left\langle Q_\mathrm{R}\Sigma Q_\mathrm{L}\Sigma^\dagger\right\rangle + \frac{1}{4}\left\langle Q_\mathrm{R}^2 + Q_\mathrm{L}^2\right\rangle\left\langle\Sigma\Sigma^\dagger\right\rangle\,, \quad (6.3)$$

and

$$c\phi^0 \to \frac{1}{2}\left\langle f\Sigma^\dagger\right\rangle\,,$$

where f denotes the spurion field $f = f^0 + \mathrm{i}f^i\tau^i$. With these assignments, the generating functional in this enlarged theory is invariant under the local $SU_\mathrm{R}(2)\times SU_\mathrm{L}(2)$ transformations

$$l_\mu \to V_\mathrm{L} l_\mu V_\mathrm{L}^\dagger - \mathrm{i}V_\mathrm{L}\partial_\mu V_\mathrm{L}^\dagger\,,$$
$$r_\mu \to V_\mathrm{R} r_\mu V_\mathrm{R}^\dagger - \mathrm{i}V_\mathrm{R}\partial_\mu V_\mathrm{R}^\dagger\,,$$
$$f \to V_\mathrm{R} f V_\mathrm{L}^\dagger\,, \quad Q_\mathrm{L} \to V_\mathrm{L} Q_\mathrm{L} V_\mathrm{L}^\dagger\,, \quad Q_\mathrm{R} \to V_\mathrm{R} Q_\mathrm{R} V_\mathrm{R}^\dagger\,;$$
$$V_\mathrm{L,R} \in SU(2)\,. \quad (6.4)$$

The effective Lagrangian $\mathcal{L}_\mathrm{eff}$ is constructed from the Goldstone boson fields, the photon field and the external sources $r_\mu, l_\mu, f, Q_\mathrm{R}$ and Q_L. The matching condition states that the Green functions in the effective theory must coincide with those in the original theory at momenta much smaller than the σ mass. At the end, one evaluates Green functions in the limit where the charge matrices become space-time independent, $Q_\mathrm{R} = Q_\mathrm{L} = \frac{1}{2}\,\mathrm{diag}(1,-1)$. Because the linear sigma model with space-time dependent spurion fields has the same symmetry as the theory that underlies the construction of the effective Lagrangian performed by Urech [1], by Meißner, Müller and Steininger [12], and by Knecht and Urech [13], we simply take over their result. For easy reference, we display this effective Lagrangian in Appendix C, adapted to the case of $SU(2)_\mathrm{R}\times SU(2)_\mathrm{L}$ which is relevant here. That Lagrangian is constructed using symmetry arguments, as a result of which the corresponding LECs are not determined. Here, we have more information at our disposal: the low-energy expansion of the (loop expanded) generating functional of the linear sigma model allows one to express these LECs through the parameters of the LσM. On the other hand, one can as well determine particular LECs by comparing physical quantities calculated in the underlying and in the effective theory. Below, we use both methods. Namely, we first verify, working out the tree approximation of the linear sigma model at low energy, that the leading term in the effective Lagrangian indeed has the structure of $\mathcal{L}^{(2)}$ displayed in (C.2). At one loop, we omit the full calculation and evaluate instead, using as examples the pion masses and the matrix elements of the vector currents determined above, several particular combinations of the LECs that occur in the low-energy effective theory. These examples already illustrate the salient features of the effective theory, in particular, the dependence on the scale in the underlying theory, as well as on the gauge parameter ξ. In addition, the meaning of the splitting between strong and electromagnetic contributions in the effective theory is clarified.

6.2 Tree level

In order to determine the structure of the effective Lagrangian at leading order in the low-energy expansion, we first perform the low-energy expansion of the Green functions in the linear σ-model at tree level. For simplicity, we stick to space-time independent charges, as in (3.1), and consider the action

$$S_\sigma = \int \mathrm{d}x \left\{ \hat{\mathcal{L}}_0 + f^{\mathrm{T}}\phi \right\}, \tag{6.5}$$

where $\hat{\mathcal{L}}_0$ denotes the Lagrangian $\mathcal{L}_0$ in (3.1) at $c = 0$, and where $f = (f^0, f^i)$. The action S_σ, evaluated at the solution to the classical equations of motion (EOM), generates the tree graphs of the linear sigma model. It therefore suffices to solve these equations in the large m_σ limit. As is the case for $e = 0$ [22, 23], the following parameterization of the ϕ field turns out to be useful:

$$\phi^A_{\mathrm{cl}} = \frac{m}{\sqrt{g}} R U^A , \qquad U^{\mathrm{T}} U = 1 , \tag{6.6}$$

where $U^{\mathrm{T}} = (U^0, U^i)$. The EOM in terms of the new fields are

$$\begin{aligned}
&\partial^\mu \partial_\mu R + R\left(U^{\mathrm{T}} d_\mu d^\mu U\right) \\
&\quad = U^{\mathrm{T}}\chi + m^2 R\left(1 - R^2\right) \\
&\qquad - e^2 m^2 R\left(c_m - 2c_g R^2\right)\left(U^{\mathrm{T}} Q^2 \cdot U\right), \\
&R\left[d_\mu d^\mu U - U\left(U^{\mathrm{T}} d_\mu d^\mu U\right)\right] \\
&\quad = \chi - U\left(U^{\mathrm{T}}\chi\right) - 2\partial_\mu R d^\mu U \\
&\qquad - e^2 m^2 R\left(c_m - c_g R^2\right)\left[Q^2 \cdot U - U\left(U^{\mathrm{T}} Q^2 \cdot U\right)\right],
\end{aligned} \tag{6.7}$$

with $\chi \doteq \frac{\sqrt{g}}{m} f$, and $d_\mu U = d_\mu \phi|_{\phi \to U}$. Analogous EOM hold for the photon field. In the following, we count the field χ as a quantity of order p^2 [22]. The solution for the radial field R becomes

$$\begin{aligned}
&R = 1 + R_2 + O\left(p^4\right), \\
&R_2 = \\
&\frac{1}{2m^2}\left\{U^{\mathrm{T}}\chi - U^{\mathrm{T}} d_\mu d^\mu U - e^2 m^2 [c_m - 2c_g]\left(U^{\mathrm{T}} Q^2 \cdot U\right)\right\}.
\end{aligned} \tag{6.8}$$

The action can finally be written in the form

$$\begin{aligned}
S_\sigma &= \int \mathrm{d}x\, \mathcal{L}^{(2)}_{\mathrm{eff}} + O\left(p^4\right), \\
\mathcal{L}^{(2)}_{\mathrm{eff}} &= F^2_{\mathrm{cl}}\left\{\frac{1}{2}\left(d_\mu U\right)^{\mathrm{T}} d^\mu U + U^{\mathrm{T}}\chi\right\} - \frac{1}{4} F_{\mu\nu,\mathrm{cl}} F^{\mu\nu}_{\mathrm{cl}} \\
&\quad - \frac{1}{2\xi}\left(\partial_\mu A^\mu_{\mathrm{cl}}\right)^2 + e^2 F^4_{\mathrm{cl}} Z_{\mathrm{cl}}\left(U^{\mathrm{T}} Q^2 \cdot U\right),
\end{aligned} \tag{6.9}$$

where

$$F^2_{\mathrm{cl}} = \frac{m^2}{g}, \qquad Z_{\mathrm{cl}} = \frac{g}{2}\left(c_g - c_m\right) \tag{6.10}$$

are the parameters of the $O\left(p^2\right)$ chiral Lagrangian, evaluated at tree level in the linear sigma model. These parameters are modified by loop contributions [22, 23].

The structure of the Lagrangian (6.9) indeed agrees with (C.2), if translated into the matrix notation used there. [In the effective Lagrangian (6.9), the fields U, A^μ_{cl} obey the EOM relevant for the LσM. To the order considered, one may, however, replace these solutions by the ones where U, A^μ_{cl} satisfy the EOM of the effective theory defined by $\mathcal{L}^{(2)}_{\mathrm{eff}}$ in (6.9).]

As already announced, we now match the expressions for several physical quantities calculated in the LσM and in the low-energy effective theory at one loop. In this manner, one may read off the values of particular linear combinations of LECs. We start the procedure by comparing the expressions for the pion masses in the underlying and in the effective theory.

6.3 Matching pion masses

We first consider the purely strong part in the pion mass, displayed in (4.15). The low-energy expansion is performed using the power counting (3.4), which amounts in this case to an expansion in the parameter c. We find that

$$\begin{aligned}
&\bar{M}^2_\pi = \\
&\bar{M}^2\left[1 - \frac{1}{32\pi^2}\frac{\bar{M}^2}{\bar{F}^2}\left(\frac{16\pi^2}{\bar{g}} - 11\ln\frac{2\bar{m}^2}{\mu^2} + \frac{22}{3} - \ln\frac{\bar{M}^2}{\mu^2}\right)\right] \\
&\quad + O\left(p^6\right),
\end{aligned} \tag{6.11}$$

where $\bar{F}^2$ and $\bar{M}^2$ are reported in Appendix E. The quantity $\bar{F}$ denotes the pion decay constant in the chiral limit, evaluated in the framework of the linear sigma model at order $\hbar$; see [22, 23], from where the expression for $\bar{F}$ is taken. We recall that these calculations are performed at one loop. The expression for $\bar{F}^2$ for instance is an expansion in $\bar{g}$ – two loops would generate terms of order $\bar{g}$, etc. The decomposition (6.11) is not unique – one may define a modified parameter $\hat{M}^2$ that differs from $\bar{M}^2$ by terms of order c^2 without modifying the structure of (6.11) – only the terms between brackets would change. Here, we have used the fact that $\bar{M}^2$ is linear in c [22, 23]. This fixes the structure of the expansion uniquely.

We may now compare (6.11) with the expansion of the pion mass in the effective theory at $e = 0$. We find for the parameters in the effective theory (see Appendices C and D)

$$\begin{aligned}
&M^2 = 2\hat{m}B = \bar{M}^2, \quad F^2 = \bar{F}^2, \\
&l^{\mathrm{r}}_3(\mu_{\mathrm{eff}}) = -\frac{1}{64\pi^2}\left(\frac{16\pi^2}{\bar{g}} - 11\ln\frac{2\bar{m}^2}{\mu^2} + \frac{22}{3} + \ln\frac{\mu^2}{\mu^2_{\mathrm{eff}}}\right), \\
&l_7 = 0.
\end{aligned} \tag{6.12}$$

Note that M^2, l_7 and F^2 are independent of the scales μ and μ_{eff} of the underlying and of the effective theory. On the other hand, the pion decay constant and the mass parameter M^2 depend on the matching scale μ_1. At one loop,

$$\frac{\mu_1}{F^2}\frac{\mathrm{d}}{\mathrm{d}\mu_1} F^2 = -2\frac{\mu_1}{M^2}\frac{\mathrm{d}}{\mathrm{d}\mu_1} M^2 = \frac{e^2\bar{g}(c_m - c_g)}{4\pi^2}. \tag{6.13}$$

The last term in this equation is proportional to the charged pion (mass)^2 in the chiral limit; see below. Using the DGMLY sum rule [24] gives

$$F(\mu_1 = 1\,\mathrm{GeV}) = F(\mu_1 = 500\,\mathrm{MeV}) - 0.1\,\mathrm{MeV}\,. \quad (6.14)$$

We now turn to the determination of some of the electromagnetic low-energy constants in the effective theory and start with the leading term Z at order p^2, which determines the charged pion mass at $c = 0$,

$$M_{\pi^+}^2 = 2e^2F^2Z + O\left(c, e^4\right). \quad (6.15)$$

From the expression (4.6) for the charged pion mass, one can derive Z (see Appendix E for the explicit expression). The quantity Z does not depend on the scale μ, whereas the dependence on the matching scale μ_1 generates a term of order e^4 in the pion mass and is disregarded.

We can also determine the linear combinations $\mathcal{K}_{\pi^0}^{\mathrm{r}}$, $\mathcal{K}_{\pi^\pm}^{\mathrm{r}}$ of the electromagnetic couplings k_i^{r} that occur in the expansion of the pion masses in the effective theory; see (D.1)–(D.3). The result is displayed in (E.2). Whereas the couplings $\mathcal{K}_{\pi^0}^{\mathrm{r}}$, $\mathcal{K}_{\pi^\pm}^{\mathrm{r}}$ in (E.2) are independent of the scale μ, they depend on the matching scale μ_1,

$$\mu_1 \frac{\mathrm{d}\mathcal{K}_{\pi^0}^{\mathrm{r}}}{\mathrm{d}\mu_1} = \mu_1 \frac{\mathrm{d}\mathcal{K}_{\pi^\pm}^{\mathrm{r}}}{\mathrm{d}\mu_1} = -\frac{Z}{4\pi^2}\,. \quad (6.16)$$

Finally, we display the neutral pion mass in the linear sigma model, properly expanded in powers of momenta, and electromagnetic corrections disentangled,

$$\begin{aligned}
M_{\pi^0}^2 &= \bar{M}_\pi^2 + e^2 M_{\pi^0}^{2,1} + O\left(e^4\right),\\
\bar{M}_\pi^2 &= M^2\left\{1 + \frac{2M^2}{F^2}\left(l_3^{\mathrm{r}} + \frac{1}{64}\ln\frac{M^2}{\mu_{\mathrm{eff}}^2}\right)\right\} + O\left(p^6\right),\\
e^2 M_{\pi^0}^{2,1} &= \frac{M^2}{16\pi^2F^2}\left\{M_{\pi^+}^2 \ln\frac{M_{\pi^+}^2}{\mu_{\mathrm{eff}}^2} - M^2\ln\frac{M^2}{\mu_{\mathrm{eff}}^2}\right\}\\
&\quad + e^2M^2\mathcal{K}_{\pi^0}^{\mathrm{r}} + O\left(p^6\right). \quad (6.17)
\end{aligned}$$

These expressions agree in form with the one displayed in (D.1) for the effective theory.

6.4 Matching the charged vector current

We start with the calculation of the matrix element of the charged vector current in the effective theory. Note that, since this matrix element is gauge dependent, we are forced to use the same gauge in the underlying and in the effective theory, otherwise the matching of these theories cannot be performed. The diagrams, contributing at one loop to this matrix element, are shown in Fig. 5. We find that[3]

$$\begin{aligned}
F_+^{\mathrm{eff}}(t) &= 1\\
&- \frac{t}{96\pi^2F^2}\left\{96\pi^2 l_6^{\mathrm{r}}(\mu_{\mathrm{eff}}) + \ln\frac{M_{\pi^+}^2}{\mu_{\mathrm{eff}}^2} - \frac{6}{t}J_c\left(t; M_{\pi^+}^2, M_{\pi^0}^2\right)\right\}\\
&+ e^2\left\{\frac{(\xi-5)}{2(4\pi)^2} + 2k_9^{\mathrm{r}}(\mu_{\mathrm{eff}}) - \frac{(3-2\xi)}{(8\pi)^2}\ln\frac{M_{\pi^+}^2}{\mu_{\mathrm{eff}}^2}\right.\\
&\quad \left. + (3-\xi)\lambda_{\mathrm{IR}}(M_{\pi^+})\right\} + O\left(p^4\right). \quad (6.18)
\end{aligned}$$

[3] In this and in the following subsection, the symbol M_{π^+} (M_{π^0}) denotes the charged (neutral) pion mass in the effective theory

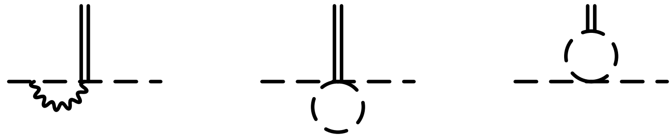

Fig. 5. One-loop contributions to the matrix element of the charged vector current in ChPT. Counterterm contributions and external lines insertions are not shown. Double, dashed and wiggle lines correspond to the external vector source, and to (charged or neutral) pions and photons, respectively

The loop function $J_c(t; m_1^2, m_2^2)$ and the infrared divergent part λ_{IR} are displayed in Appendix B. The first line in (6.18) reproduces the function $\tilde{f}_+(t)$ introduced in [5] [in the limit where the strange quark mass is taken to be large in $\tilde{f}_+(t)$].The matching of F_+^{eff} to F_+ in (5.4) enables one to read off the values of the strong and electromagnetic LECs. At tree level, the matrix elements are equal to one in both cases. At one-loop accuracy, one may safely use the tree-level relations $M_{\pi^{+,0}}^2 = m_{\pi^{+,0}}^2 + O(\hbar, p^4)$ and $F^2 = m^2/g + O(\hbar)$ everywhere in the form factor (6.18). The matching condition simplifies to

$$\begin{aligned}
l_6^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= -\frac{1}{96\pi^2}\ln\frac{2\bar{m}^2}{\mu_{\mathrm{eff}}^2} + \frac{11}{36}\frac{1}{16\pi^2}\,,\\
k_9^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= \frac{1}{64\pi^2}\left(\frac{3}{2} - \xi\right)\left(1 + \ln\frac{\mu^2}{\mu_{\mathrm{eff}}^2}\right). \quad (6.19)
\end{aligned}$$

The value for $l_6^{\mathrm{r}}(\mu_{\mathrm{eff}})$ agrees with that from [22, 23]. As expected, $l_6^{\mathrm{r}}(\mu_{\mathrm{eff}})$ does not depend on the underlying scale μ, in contrast to $k_9^{\mathrm{r}}(\mu_{\mathrm{eff}})$. The whole μ-dependence of the matrix element of the charged vector current in the underlying theory is generated by the coupling $k_9^{\mathrm{r}}(\mu_{\mathrm{eff}})$,

$$\begin{aligned}
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}F_+^{\mathrm{eff}}(t) &= \frac{e^2}{32\pi^2}(3-2\xi)\,,\\
\mu\frac{\mathrm{d}}{\mathrm{d}\mu}k_9^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= \frac{1}{64\pi^2}(3-2\xi)\,. \quad (6.20)
\end{aligned}$$

We note that both, the μ- as well as the ξ-dependence of $k_9^{\mathrm{r}}(\mu_{\mathrm{eff}})$, are unambiguously determined by the underlying theory, and reflect the fact that the matrix element of the charged current is subject to ambiguities. One may, however, imagine a situation where the LσM is embedded in a larger theory with electrons and neutrinos, and calculate the S-matrix element corresponding to the pion β decay in such a theory. The S-matrix element, calculated in the corresponding enlarged effective theory, then contains LECs from the lepton sector, which cancel the scale and gauge dependence of $k_9^{\mathrm{r}}(\mu_{\mathrm{eff}})$ (here, we neglect the problem with infrared divergences). The above example shows that the scale and gauge dependence in various LECs can be strongly correlated. In particular, the conventional dimensional estimate can hold only for the invariant combinations of LECs, which enter physical quantities. Finally, we mention that at this order of the perturbation expansion, there is no dependence on the scale μ_1 in $l_6^{\rm r}(\mu_{\rm eff})$, nor in $k_9^{\rm r}(\mu_{\rm eff})$.

6.5 Matching the neutral vector current

The analog of (5.6) in the effective theory reads

$$\begin{aligned}
&\Phi^{\rm eff}(t) = 1 \\
&-\frac{t}{96\pi^2F^2}\left\{96\pi^2 l_6^{\rm r}(\mu_{\rm eff}) + \ln\frac{M_{\pi^+}^2}{\mu_{\rm eff}^2} + \frac{1}{3} + \sigma^2 K\left(t/M_{\pi^+}^2\right)\right\} \\
&+\frac{e^2}{16\pi^2}\left\{\left(\frac{t}{M_{\pi^+}^2} - 2\right) G\left(t/M_{\pi^+}^2\right) + \frac{\sigma^2}{3}K\left(t/M_{\pi^+}^2\right) - \frac{2}{9}\right. \\
&\left.+2\frac{\left(2M_{\pi^+}^2 - t\right)K\left(t/M_{\pi^+}^2\right) - t}{t - 4M_{\pi^+}^2}\left[32\pi^2\lambda_{\rm IR}(M_{\pi^+}) - 1\right]\right\} \\
&+O\left(p^4\right), \\
&\sigma^2 = 1 - \frac{4M_{\pi^+}^2}{t}\,. \qquad (6.21)
\end{aligned}$$

The loop functions G and K and the infrared divergent part $\lambda_{\rm IR}$ are displayed in Appendix B. We note that $\Phi^{\rm eff}(t)$ is identical to the form factor $F_\pi^V(t)$ displayed in (3.7) of [25], provided that one normalizes $\bar{l}_6$ used there to the charged pion mass. In that paper, the infrared singularities are regularized by introducing a small photon mass m_γ. The correspondence rule with dimensional regularization is given in Appendix B.

The matching of $\Phi^{\rm eff}(t)$ to the corresponding expression in the LσM does not lead to any additional information, since $l_6^{\rm r}(\mu_{\rm eff})$ was already determined by the charged current matrix element. Non-trivial information can be extracted from matching the matrix elements at $t = 0$. We find

$$F_0^{\rm eff}(0) = 1 - e^2\left[8h_2^{\rm r}(\mu_{\rm eff}) - \frac{1}{48\pi^2}\left(1 + \ln\frac{M_{\pi^+}^2}{\mu_{\rm eff}^2}\right)\right], \qquad (6.22)$$

where $h_2^{\rm r}(\mu_{\rm eff})$ denotes the high-energy constant from the $O(p^4)$ Lagrangian [22]; see Appendix C. According to (6.22), the form factor is not normalized to 1 at zero momentum transfer, in contrast to the statement made in [25].

Matching of (6.22) and (5.7) enables one to determine $h_2^{\rm r}(\mu_{\rm eff})$,

$$h_2^{\rm r}(\mu_{\rm eff}) = \frac{1}{24}\frac{1}{16\pi^2}\left(\ln\frac{\mu^2}{\mu_{\rm eff}^2} + 1\right). \qquad (6.23)$$

This expression agrees with the corresponding expression from [22, 23], if the $\overline{\rm MS}$ renormalization scheme is used there to remove the divergences in the two-point function of two vector sources. As expected, at the level of the effective theory, the dependence on the underlying scale μ appears in the effective couplings, which is $h_2^{\rm r}(\mu_{\rm eff})$ in the present case.

7 Comparison with other approaches

The scale and gauge dependence of electromagnetic LECs has been discussed in the literature before [7, 8, 10], in the framework of QCD + γ. Here, we compare our approach with those works. The comparison with the procedure of [8] is complicated by the fact that these authors use different models to describe the physics at different momenta, and introduce several scales to separate momentum regimes. On the other hand, the prescription for the splitting of strong and electromagnetic effects considered in Moussallam's work [10] is the same as in [8], and we therefore stick to a comparison with that article for simplicity.

7.1 Pion mass in QCD + γ

In order to illustrate the treatment followed by Moussallam, we first consider the quark mass expansion of the charged pion mass in the effective theory of QCD + γ in $SU(3)\times SU(3)$, because Moussallam's article refers to this framework. This expansion has been worked out by Urech [1] up to and including terms of order p^4, e^2p^2 in the limit where $m_u = m_d$. We relax this condition and find

$$\begin{aligned}
M_{\pi^+}^2 &= (m_u + m_d)B_0 + 2e^2 Z_0 F_0^2 \qquad (7.1)\\
&\quad + e^2\left(m_u K_u^{\rm r} + m_d K_d^{\rm r}\right)B_0 + L_{p^4} + O\left(p^6\right).
\end{aligned}$$

Here, B_0, F_0, Z_0 stand for B, F, Z evaluated at $m_s = 0$, and $K_q^{\rm r}$ are linear combinations of the $SU(3)\times SU(3)$ analogues of the electromagnetic LECs $k_i^{\rm r}$. The symbol L_{p^4} denotes contributions from loops and from strong counterterms at order $m_q^2, e^2 m_q$ – the explicit expressions for these terms are not needed in the following. The point we wish to make here is the fact that, according to [10], the LECs $K_q^{\rm r}$ depend on the QCD + γ scale μ as follows:

$$\mu\frac{\rm d}{{\rm d}\mu}\left(K_u^{\rm r}, K_d^{\rm r}\right) = \frac{1}{24\pi^2}(4, 1) + O\left(g_s^2\right), \qquad (7.2)$$

where g_s denotes the strong coupling constant. On the other hand, Z_0 and F_0 are scale independent [10]. As the pion mass is scale independent as well, one has

$$\begin{aligned}
\mu\frac{\rm d}{{\rm d}\mu}(m_u + m_d)B_0 &= -\frac{e^2}{24\pi^2}(4m_u + m_d)B_0 \\
&\quad + O\left(m_q e^2 g_s^2, m_q e^4\right). \qquad (7.3)
\end{aligned}$$

Note that the scale dependence of the loop contributions L_{p^4} is then of order $e^2 m_q^2, e^4 m_q$ and thus beyond the accuracy considered here. The relation (7.3) may be compared with the running of the quark masses in QCD + γ,

$$\begin{aligned}
\mu\frac{\rm d}{{\rm d}\mu}(m_u, m_d) &= -\frac{e^2}{24\pi^2}(4m_u, m_d) \\
&\quad + O\left(m_q g_s^2, m_q e^2 g_s^2, m_q e^4\right). \qquad (7.4)
\end{aligned}$$

We conclude that, in this framework, some of the parameters in the effective theory of the strong Lagrangian run with the β-function of QCD $+\, \gamma$.[4]

The splitting into strong and electromagnetic effects advocated in the present work has a different structure. Indeed, what we call *strong* and *electromagnetic* parts of physical quantities are independent of the scale μ in the underlying theory, and the scale dependence of the LECs differs from the one found in [8, 10]. In order to illustrate the point, we consider again the LσM.

7.2 Splitting in the linear sigma model

The expression of the neutral pion mass has been worked out in Sect. 4. We write the result (4.6) for the neutral pion mass in the form

$$\begin{aligned} M_{\pi^0}^2 &= f_0 + e^2 f_1 + O\left(e^4, p^6\right), \\ f_0 &= m_{\pi^0}^2 \left\{1 + \frac{g}{m_\sigma^2}\left(V_0 + L_{\pi^0}\right)\right\}, \\ e^2 f_1 &= 2 m_{\pi^0}^2 \frac{g}{m_\sigma^2}\left\{L_{\pi^+} - L_{\pi^0}\right\}. \end{aligned} \quad (7.5)$$

Since the physical mass is scale independent, one has

$$\mu \frac{\mathrm{d} f_0}{\mathrm{d}\mu} = -e^2 \mu \frac{\mathrm{d} f_1}{\mathrm{d}\mu}. \quad (7.6)$$

Consider now the splitting of electromagnetic and strong effects. In the language of [10], f_0 ($e^2 f_1$) is the *strong* (*electromagnetic*) part of the physical mass. Both, the strong and the electromagnetic parts of the mass, are μ-dependent in this case. One may again work out the low-energy representation of $M_{\pi^0}^2$ and identify the low-energy constants in this language. For the strong part, one finds the expressions displayed in (6.11)–(6.12), with $(\bar{g}, \bar{m}^2) \to (g, m^2)$, whereas the electromagnetic LECs are collected in

$$\mathcal{K}_{\pi^0}^{\mathrm{r}} = \frac{(c_g - c_m) g}{16\pi^2}\left(\ln \frac{\mu_{\mathrm{eff}}^2}{\mu^2} - 1\right); \quad \text{splitting according to [10]}. \quad (7.7)$$

Here, the μ-dependence of $\mathcal{K}_{\pi^0}^{\mathrm{r}}$ shows up. As is the case in QCD $+\, \gamma$ discussed above, this scale dependence of the electromagnetic part is canceled by the corresponding scale dependence of the strong part in the framework of [10].

In our framework, the strong part is given by

$$\bar{M}_{\pi^0}^2 = f_0 \Big|_{g=\bar{g}, m=\bar{m}, c=\bar{c}}, \quad (7.8)$$

where the couplings $\bar{g}, \bar{m}$ run with the strong part alone; see the discussion in earlier sections. The difference $M_{\pi^0}^2 - \bar{M}_{\pi^0}^2$ is called *electromagnetic correction* in this article. Both, the strong and the electromagnetic parts, are μ-independent. The electromagnetic LEC $\mathcal{K}_{\pi^0}^{\mathrm{r}}$, calculated using our matching procedure, is displayed in (E.2). We note that the μ-dependence of $\mathcal{K}_{\pi^0}^{\mathrm{r}}$ in (7.7) is the same as the μ_1-dependence in (E.2). One can show that such a correspondence exists for all quantities that are μ-independent. On the other hand, it does not hold anymore e.g. in the case of the charged form factor, whose matrix elements are μ-dependent.

We add a remark concerning the scale dependence of the electromagnetic LECs as determined in [8, 10]. Since both references use the same procedure, the scale dependences found in [8, 10] should agree. On the other hand, this is not the case e.g. for the coupling K_{12}; see the remark at the end of Sect. 3.4 in [10].

Recently, disentangling electromagnetic contributions has become of relevance in connection with the anomalous magnetic moment of the muon, and with a precise determination of the CKM matrix elements. As an example, we quote the work of Cirigliano et al. [5, 6]. The authors use a splitting in the framework of the effective theory of the standard model (see e.g. (3.4) in [5]). The dependence of the couplings on the scale of the underlying theory is not investigated. As these calculations are not in close connection with the issues addressed here, we do not compare them with the present framework.

8 Summary and conclusions

(i) In several applications of ChPT, one is forced to purify measured matrix elements from electromagnetic interactions, in order to extract what is usually called a *hadronic* quantity. As a simple example, we mentioned in the introduction the mass difference of charged and neutral kaons in pure QCD, a quantity that enters the calculation of the decay $\eta \to 3\pi$ in the effective low-energy theory of QCD. It is well known that, due to the ultraviolet divergences generated by photon loops, a purification from electromagnetic effects cannot be performed in a unique manner. This issue has been discussed earlier in [7, 8, 10]. Our article is devoted to a more detailed analysis of this problem.
(ii) In order to achieve the splitting in a systematic manner, we propose to match the parameters of two theories – with and without electromagnetic interactions – at a given scale μ_1, referred to as the matching scale. This enables one to analyse the separation of strong and electromagnetic contributions in a transparent manner. We first study in a Yukawa-type model the dependence of strong and electromagnetic contributions on the matching scale μ_1. In a second step, we investigate this splitting in the linear sigma model (in the presence of virtual photons) at one-loop order, and consider in some detail the construction of the corresponding low-energy effective Lagrangian. The effective theory exactly implements the splitting of electromagnetic and strong interactions carried out in the underlying theory, provided that the LECs are properly chosen.
(iii) In our prescription for disentangling electromagnetic effects, the parameters of the effective Lagrangian in the strong sector are expressed through the parameters of the underlying theory in its strong sector ($\bar{g}, \bar{m}, \chi$ in the case

[4] This discussion parallels the one for the quantity $\epsilon_2 + e^2 \delta_2$ in Sect. 4 of [16]

of the linear sigma model). Apart from the μ_1-dependence, the LECs of the effective theory contain the full information about the scale and gauge dependence of Green functions in the underlying theory, which arises when the electromagnetic interactions are switched on. We have studied this phenomenon by considering the matrix element of the neutral and charged vector currents in the linear sigma model for illustration.
(iv) An example of the splitting in the effective theory is provided by the low-energy expansion of the neutral pion mass, which reads

$$M_{\pi^0}^2 = \bar{M}_\pi^2 + e^2 M_{\pi^0}^{2,1} + O\left(e^4\right), \tag{8.1}$$

where $\bar{M}_\pi^2$ $\left(e^2 M_{\pi^0}^{2,1}\right)$ denotes the strong (electromagnetic) part. Both parts depend on μ_1, in such a manner that the physical mass is μ_1 independent. The result (8.1) is the analogue of the splitting that one needs to perform for the kaon masses in the calculation of the decay width $\Gamma_{\eta\to 3\pi}$.
(v) A second example is given by the pion decay constant in the chiral limit, where we find that

$$F(\mu_1 = 1\,\mathrm{GeV}) = F(\mu_1 = 500\,\mathrm{MeV}) - 0.1\,\mathrm{MeV} \tag{8.2}$$

in the framework of the linear sigma model as the underlying theory. Note that this scale dependence is of the same order of magnitude as the experimental uncertainty for the pion decay constant quoted by the PDG [27].
(vi) It would be of interest to study in a next step the low-energy effective theory of QCD + γ along these lines. Once the dependence of the LECs on the QCD scale μ and on the matching scale μ_1 is determined, a calculation of electromagnetic corrections in the framework of the effective theory would reflect the corresponding splitting in QCD + γ.

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A Charge matrices

We use the following notation for the charge matrices.
(1) For Yukawa theory we have

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}; \tag{A.1}$$

see (2.4).
(2) For the linear sigma model we use a 4×4 matrix Q_{AB}, see Sect. 3.1. The only non-zero entries are $Q_{12} = -Q_{21} = -1$. For the matrix, we use the same symbol as in (A.1), because it is clear from the context which version is meant.
(3) In Appendix C we use (A.1), together with

$$\hat{Q} = \frac{1}{2}\,\mathrm{diag}\,(1,-1)\,. \tag{A.2}$$

B Ultraviolet and infrared divergences, loop integrals

B.1 Divergences

Throughout this paper, we tame both, ultraviolet and infrared divergences, with dimensional regularization. Ultraviolet divergences are proportional to

$$\lambda(\mu) = \frac{\mu^{d-4}}{16\pi^2}\left(\frac{1}{d-4} - \frac{1}{2}\left(\Gamma'(1) + \ln 4\pi\right)\right). \tag{B.1}$$

As usual, d denotes the dimension of space-time, and μ is the renormalization scale. In the text, we also use the infrared divergent quantity

$$\lambda_{\mathrm{IR}}(m) = \frac{m^{d-4}}{16\pi^2}\left(\frac{1}{d-4} - \frac{1}{2}\left(\Gamma'(1) + \ln 4\pi\right)\right). \tag{B.2}$$

Here, m is a mass parameter, identified in the text with the pion mass. Infrared singularities may instead be tamed by introducing a small photon mass. The correspondence rule is

$$\lambda_{\mathrm{IR}}(m) \to -\frac{1}{32\pi^2}\left(\ln\frac{m_\gamma^2}{m^2} + 1\right). \tag{B.3}$$

B.2 Meson loops

Here, we collect the meson loop functions used in the text. We have

$$G(y) = \int_0^1 \frac{\mathrm{d}x}{1 - x(1-x)y}\,\ln\left(1 - x(1-x)y\right),$$

$$K(y) = \int_0^1 \mathrm{d}x\,\ln\left(1 - x(1-x)y\right), \tag{B.4}$$

$$J_c\left(t; m_1^2, m_2^2\right) = \frac{m_2^2}{2}\,\ln\frac{m_1^2}{m_2^2} + \int_0^1 \mathrm{d}x\, g(x;t)\ln\frac{g(x;t)}{m_1^2}\,,$$

where

$$g(x;t) = x m_1^2 + (1-x) m_2^2 - x(1-x)t\,. \tag{B.5}$$

At equal mass, one has

$$J_c\left(t; m^2, m^2\right) = -\frac{1}{6}\left(t - 4m^2\right) K\left(t/m^2\right) - \frac{t}{18}. \tag{B.6}$$

C Effective theory

We display the effective Lagrangian for $SU(2)_{\rm R} \times SU(2)_{\rm L}$ in the presence of virtual photons [12, 13]. It serves at the same time as the effective Lagrangian for the linear sigma model, as we discussed in Sect. 6. The Lagrangian has the form

$$\mathcal{L} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots \tag{C.1}$$

C.1 Leading order

The leading order Lagrangian is [1]

$$\begin{aligned}\mathcal{L}^{(2)} = & \frac{F^2}{4}\left\langle d^\mu U^+ d_\mu U + \chi^+ U + U^+\chi\right\rangle - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ & -\frac{1}{2\xi}\left(\partial^\mu A_\mu\right)^2 + ZF^4 e^2\left\langle \hat{Q}U\hat{Q}U^+\right\rangle,\end{aligned} \tag{C.2}$$

with $U \in SU(2)$, and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad d_\mu U = \partial_\mu U - {\rm i}R_\mu U + {\rm i}UL_\mu,$$
$$\chi = 2B(s + {\rm i}p)\ ,$$

and

$$R_\mu = v_\mu + eA_\mu\hat{Q} + a_\mu\ , L_\mu = v_\mu + eA_\mu\hat{Q} - a_\mu\ ,$$
$$\hat{Q} = \frac{1}{2}\,{\rm diag}\,(1, -1)\,.$$

The symbol $\langle\ldots\rangle$ denotes the trace in flavour space. The external fields v_μ, a_μ, p and s are given by

$$v_\mu = v_\mu^i\frac{\tau^i}{2}, \quad a_\mu = a_\mu^i\frac{\tau^i}{2},$$
$$s = s^0\mathbf{1} + s^i\tau^i, \quad p = p^0\mathbf{1} + p^i\tau^i,$$

where τ^i denote the Pauli matrices. Note that the left – and right-handed external fields are traceless. The mass matrix of the two light quarks is contained in s,

$$s = \mathcal{M} + \ldots, \quad \mathcal{M} = {\rm diag}\,(m_u, m_d)\,. \tag{C.3}$$

The quantity ξ denotes the gauge fixing parameter, and the parameters F, B and Z are the three low-energy coupling constants occurring at leading order.

C.2 Next-to-leading order

The next-to-leading order Lagrangian reads

$$\mathcal{L}^{(4)} = \mathcal{L}_{p^4} + \mathcal{L}_{p^2e^2} + \mathcal{L}_{e^4}. \tag{C.4}$$

The Lagrangian at order p^4 was constructed in [13,22,26],

$$\begin{aligned}\mathcal{L}_{p^4} = & \frac{l_1}{4}\left\langle d^\mu U^+ d_\mu U\right\rangle^2 + \frac{l_2}{4}\left\langle d^\mu U^+ d^\nu U\right\rangle\left\langle d_\mu U^+ d_\nu U\right\rangle \\ & + \frac{l_3}{16}\left\langle \chi^+ U + U^+\chi\right\rangle^2 + \frac{l_4}{4}\left\langle d^\mu U^+ d_\mu\chi + d^\mu\chi^+ d_\mu U\right\rangle \\ & + l_5\left\langle R_{\mu\nu}UL^{\mu\nu}U^+\right\rangle \\ & + \frac{{\rm i}l_6}{2}\left\langle R_{\mu\nu}d^\mu U d^\nu U^+ + L_{\mu\nu}d^\mu U^+ d^\nu U\right\rangle \\ & - \frac{l_7}{16}\left\langle \chi^+ U - U^+\chi\right\rangle^2 \\ & + \frac{1}{4}(h_1 + h_3)\left\langle \chi^+\chi\right\rangle + \frac{1}{2}(h_1 - h_3)Re(\det\chi) \\ & - \frac{1}{2}(l_5 + 4h_2)\left\langle R_{\mu\nu}R^{\mu\nu} + L_{\mu\nu}L^{\mu\nu}\right\rangle\ ,\end{aligned} \tag{C.5}$$

with right- and left-handed field strengths defined as

$$I_{\mu\nu} = \partial_\mu I_\nu - \partial_\nu I_\mu - {\rm i}\left[I_\mu, I_\nu\right]\ , \quad I = R, L\,.$$

The coefficient of $\langle R_{\mu\nu}R^{\mu\nu} + L_{\mu\nu}L^{\mu\nu}\rangle$ in (C.5) differs from the one in [13]; see [26]. The most general list of counterterms occurring at order p^2e^2 was given in [12,13]; see also the comments in Sect. 3 of [13] for a comparison of the two works. Here, we use the notation of [13]. [In the present case, the charge matrix is traceless. Therefore, the effective Lagrangian could be written in terms of $\hat{Q}$ introduced above. On the other hand, in [13], the Lagrangian is written with a charge matrix that is not traceless. This amounts to a change of basis in the counterterms, except for the term proportional to k_7, which does not occur for a traceless charge matrix. In order to have the standard notation, we use the notation of [13] and drop the term proportional to k_7 in the Lagrangian, as well as in the expressions for the pion masses in Sect. D.] We have

$$\begin{aligned}\mathcal{L}_{p^2e^2} = & F^2e^2\{k_1\left\langle d^\mu U^+ d_\mu U\right\rangle\left\langle Q^2\right\rangle \\ & + k_2\left\langle d^\mu U^+ d_\mu U\right\rangle\left\langle QUQU^+\right\rangle \\ & + k_3\left(\left\langle d^\mu U^+ QU\right\rangle\left\langle d_\mu U^+ QU\right\rangle + \left\langle d^\mu UQU^+\right\rangle\left\langle d_\mu UQU^+\right\rangle\right) \\ & + k_4\left\langle d^\mu U^+ QU\right\rangle\left\langle d_\mu UQU^+\right\rangle + k_5\left\langle \chi^+ U + U^+\chi\right\rangle\left\langle Q^2\right\rangle \\ & + k_6\left\langle \chi^+ U + U^+\chi\right\rangle\left\langle QUQU^+\right\rangle \\ & + k_8\left\langle\left(\chi U^+ - U\chi^+\right)QUQU^+ + \left(\chi^+ U - U^+\chi\right)QU^+QU\right\rangle \\ & + k_9\left\langle d_\mu U^+\left[c_{\rm R}^\mu Q, Q\right]U + d_\mu U\left[c_{\rm L}^\mu Q, Q\right]U^+\right\rangle \\ & + k_{10}\left\langle c_{\rm R}^\mu QUc_{{\rm L}\mu}QU^+\right\rangle + k_{11}\left\langle c_{\rm R}^\mu Qc_{{\rm R}\mu}Q + c_{\rm L}^\mu Qc_{{\rm L}\mu}Q\right\rangle\},\end{aligned} \tag{C.6}$$

and

$$\begin{aligned}\mathcal{L}_{e^4} = F^4e^4\Big\{ & k_{12}\left\langle Q^2\right\rangle^2 + k_{13}\left\langle QUQU^+\right\rangle\left\langle Q^2\right\rangle \\ & + k_{14}\left\langle QUQU^+\right\rangle^2\Big\},\end{aligned} \tag{C.7}$$

where

$$c_I^\mu Q = -{\rm i}\left[I^\mu, Q\right], \quad I = R, L\,, \tag{C.8}$$

and where Q is given in (A.1). The renormalization of the low-energy constants of $\mathcal{L}^{(4)}$ is

$$
\begin{aligned}
l_i &= l_i^{\mathrm{r}}(\mu_{\mathrm{eff}}) + \gamma_i \left[\lambda(\mu_{\mathrm{eff}}) - \frac{1}{32\pi^2}\right], \quad i = 1, \ldots, 7 \,, \\
h_i &= h_i^{\mathrm{r}}(\mu_{\mathrm{eff}}) + \delta_i \left[\lambda(\mu_{\mathrm{eff}}) - \frac{1}{32\pi^2}\right], \quad i = 1,\, 2,\, 3 \,, \\
k_i &= k_i^{\mathrm{r}}(\mu_{\mathrm{eff}}) + \sigma_i \left[\lambda(\mu_{\mathrm{eff}}) - \frac{1}{32\pi^2}\right], \quad i = 1, \ldots, 14 \,.
\end{aligned}
\tag{C.9}
$$

The coefficients γ_i and δ_i were computed in [22], and the coefficients σ_i are calculated in [13] for the case $\xi = 1$. For the coupling k_9 in a generic gauge we find

$$
k_9 = k_9^{\mathrm{r}}(\mu_{\mathrm{eff}}) + \frac{3 - 2\xi}{4}\left[\lambda(\mu_{\mathrm{eff}}) - \frac{1}{32\pi^2}\right] \,. \tag{C.10}
$$

It agrees at $\xi = 1$ with the expression given in [13].

D Pion masses in ChPT

For convenience, we display the low-energy expansion of the charged and neutral pion mass at next-to-leading order in the framework of $SU(2)_{\mathrm{R}} \times SU(2)_{\mathrm{L}}$ [13]. We drop terms proportional to k_7, as discussed in Appendix C.

$$
\begin{aligned}
M_{\pi^0}^2 = M^2 \Bigg\{ & 1 + 2\frac{M^2}{F^2}\left(l_3^{\mathrm{r}} + \frac{1}{64\pi^2}\ln\frac{M^2}{\mu_{\mathrm{eff}}^2}\right) \\
& + \frac{1}{16\pi^2 F^2}\left(M_{\pi^+}^2 \ln\frac{M_{\pi^+}^2}{\mu_{\mathrm{eff}}^2} - M^2 \ln\frac{M^2}{\mu_{\mathrm{eff}}^2}\right) \\
& + e^2 \mathcal{K}_{\pi^0}^{\mathrm{r}} \Bigg\} \\
& -2\frac{B^2}{F^2}(m_d - m_u)^2 l_7 + O\left(p^6\right) ,
\end{aligned}
\tag{D.1}
$$

$$
\begin{aligned}
M_{\pi^+}^2 = M^2 \Bigg\{ & 1 + 2\frac{M^2}{F^2}\left(l_3^{\mathrm{r}} + \frac{1}{64\pi^2}\ln\frac{M^2}{\mu_{\mathrm{eff}}^2}\right) \\
& + e^2\left(\mathcal{K}_{\pm}^{\mathrm{r}} + \frac{1}{4\pi^2}\right)\Bigg\} + 2e^2 Z F^2 \\
& - \frac{e^2(3 + 4Z)M_{\pi^+}^2}{16\pi^2}\ln\frac{M_{\pi^+}^2}{\mu_{\mathrm{eff}}^2} + O\left(e^4, p^6\right) ,
\end{aligned}
\tag{D.2}
$$

where

$$
\begin{aligned}
M^2 &= 2\hat{m}B \,, \\
\mathcal{K}_{\pi^0}^{\mathrm{r}} &= -\frac{20}{9}\left[k_1^{\mathrm{r}} + k_2^{\mathrm{r}} - \frac{9}{10}\left(2k_3^{\mathrm{r}} - k_4^{\mathrm{r}}\right) - k_5^{\mathrm{r}} - k_6^{\mathrm{r}}\right] , \\
\mathcal{K}_{\pi^\pm}^{\mathrm{r}} &= -\frac{20}{9}\left[k_1^{\mathrm{r}} + k_2^{\mathrm{r}} - k_5^{\mathrm{r}} - \frac{1}{5}\left(23k_6^{\mathrm{r}} + 18k_8^{\mathrm{r}}\right)\right] .
\end{aligned}
\tag{D.3}
$$

E Matching LECs

For easy reference, we collect in this appendix the parameters of the low-energy effective Lagrangian, that we have determined to one loop in this article. The scale μ denotes the running scale in the linear sigma model; see Sect. 3.2. The barred quantities $\bar{g}, \bar{m}$ indicate the running couplings in the LσM at $e = 0$. They depend on the matching scale μ_1. The running scale in the effective theory is denoted by μ_{eff}. Finally, ξ denotes the gauge parameter in the photon propagator.

Strong LECs

At one-loop order, one has

$$
\begin{aligned}
M^2 &= \bar{M}^2 , \qquad F^2 = \bar{F}^2 , \\
\bar{M}^2 &= c\left(\frac{\bar{g}}{\bar{m}^2}\right)^{1/2}\left(1 + \frac{3\bar{g}}{16\pi^2}\ln\frac{2\bar{m}^2}{\mu^2} - \frac{\bar{g}}{4\pi^2}\right) , \\
\bar{F}^2 &= \frac{\bar{m}^2}{\bar{g}}\left(1 - \frac{3\bar{g}}{8\pi^2}\ln\frac{2\bar{m}^2}{\mu^2} + \frac{7\bar{g}}{16\pi^2}\right) , \\
l_3^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= -\frac{1}{64\pi^2}\left(\frac{16\pi^2}{\bar{g}} - 11\ln\frac{2\bar{m}^2}{\mu^2} + \frac{22}{3} + \ln\frac{\mu^2}{\mu_{\mathrm{eff}}^2}\right) , \\
l_6^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= -\frac{1}{96\pi^2}\ln\frac{2\bar{m}^2}{\mu_{\mathrm{eff}}^2} + \frac{11}{36}\frac{1}{16\pi^2} , \\
l_7 &= 0 , \\
h_2^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= \frac{1}{24}\frac{1}{16\pi^2}\left(\ln\frac{\mu^2}{\mu_{\mathrm{eff}}^2} + 1\right) .
\end{aligned}
\tag{E.1}
$$

Electromagnetic LECs

At one-loop order, we find

$$
\begin{aligned}
Z &= \frac{1}{2}\bar{g}\Bigg\{c_g\left(1 + \frac{3\bar{g}}{4\pi^2}\ln\frac{2\bar{m}^2}{\mu^2} - \frac{7\bar{g}}{8\pi^2}\right) \\
&\qquad - c_m\left(1 + \frac{\bar{g}}{2\pi^2}\ln\frac{2\bar{m}^2}{\mu^2} - \frac{5\bar{g}}{8\pi^2}\right)\Bigg\} , \\
k_9^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= \frac{1}{64\pi^2}\left(\frac{3}{2} - \xi\right)\left(1 + \ln\frac{\mu^2}{\mu_{\mathrm{eff}}^2}\right) , \\
16\pi^2\mathcal{K}_{\pi^\pm}^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= (3 + 4Z)\ln\frac{2m^2}{\mu_{\mathrm{eff}}^2} + 2Z\ln\frac{2m^2}{\mu_1^2} + 3 \\
&\qquad - \frac{5Z}{3} - 3\ln\frac{2m^2}{\mu^2} \\
&\qquad + c_g\left\{16\pi^2 + 4g\left(2\ln\frac{2m^2}{\mu^2} - 1\right)\right\} , \\
16\pi^2\mathcal{K}_{\pi^0}^{\mathrm{r}}(\mu_{\mathrm{eff}}) &= 2Z\left(\ln\frac{\mu_{\mathrm{eff}}^2}{\mu_1^2} - 1\right) .
\end{aligned}
\tag{E.2}
$$

The expression for F^2 was taken from [22,23]. Further, the expressions for M^2, l_3^r, l_6^r, l_7, h_2^r agree with the ones determined in those papers.

F Glossary of mass definitions in the LσM and in the effective theory

In this appendix we collect the definitions of various mass parameters used in the text. The following notation is used in the LσM as well as in the effective theory,

$$M^2_{\pi^{0,+}} = \bar{M}^2_\pi + e^2 M^{2,1}_{\pi^{0,+}} + O(e^4) \ ,$$

with

$M^2_{\pi^{0,+}}$	-	physical (mass)^2 of the pions
$\bar{M}^2_\pi$	-	strong part of $M^2_{\pi^{0,+}}$
$e^2 M^{2,1}_{\pi^{0,+}}$	-	electromagnetic part of $M^2_{\pi^{0,+}}$.

The mass parameter in the effective theory is the standard one,

$$M^2 = 2\hat{m}B \ .$$

In the LσM, we use

m^2	-	mass parameter of the $O(4)$ symmetric part in $\mathcal{L}_\sigma$
δm^2	-	isospin breaking mass parameter in $\mathcal{L}_\sigma$
$m^2_{\pi^{0,+}}$	-	tree level (mass)^2 of the pions
m^2_σ	-	tree level (mass)^2 of the heavy particle
$\bar{m}^2,\ \bar{m}^2_\pi$	-	strong part of m^2, $m^2_{\pi^{0,+}}$
$\bar{M}^2$	-	term of order p^2 in $\bar{M}^2_\pi$.

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